TORSIONAL VIBRATIONS OF FINITE HOLLOW POROELASTIC CIRCULAR CYLINDERS

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(*Received 7 June*, 1977)

In this paper, the problem of torsional vibrations of a homogeneous, isotropic poroelastic hollow cylinder of finite length is solved. Results of solid cylinder of finite length and classical theory of elasticity are obtained as a particular case. Frequency is calculated numerically for different materials and presented in tables.

An understanding of the free vibration of any beam is a prerequisite to the understanding of its response in forced vibration. Propagation of elastic waves and vibrations in circular rods of uniform cross-section has been extensively studied in the treatises of Kolsky[1] and Love[2]. The theory of wave propagation in solid circular rods was extended by Ghosh[3,4] to consider wave propagation along hollow cylinders. Later this is developed by Stanisic and Osburn[5] to study torsional vibrations for inhomogeneous anisotropic hollow cylinders.

The study of torsional vibrations of an elastic solid is important in several fields e.g., soil mechanics, transmission of power through shafts with flange at the end as integral part of the shaft. It is now recognised that virtually no high speed equipment can be properly designed without obtaining solution to what are essentially lateral or torsional vibration problems. Examples of torsional vibrations are vibrations in gear train and motorpump shafts. Thus, from engineering point of view the study of torsional vibrations has greater interest. In this paper problem of torsional vibrations of a homogeneous, isotropic poroelastic hollow cylinder of finite length is attempted and it is shown that solid cylinder of finite length is a particular case of it. Neglecting fluid effects, results of elastic case are obtained. The expression for frequency is expressed in non-dimensional form suitable for numerical computation and calculated for different poroelastic materials discussed by Biot[6], Nowinski and Davis[7] and are presented in tables. The material given by Nowinski and Davis[7] is bone. It can be considered as a poroelastic material[8] as a model of bone in a sense that osseous tissue is considered as a perfectly elastic solid and fluid substances filling the cavities as viscous compressible fluid.

FORMULATION AND SOLUTION OF THE PROBLEM

Torsional vibrations in a finite hollow circular cylinder of uniform cross-section of a homogeneous isotropic poroelastic material is taken up. Let the length of cylinder be l_1 and outer and inner radii be a_1 and a_2 respectively. For torsional vibrations the displacement components are

$$\begin{aligned} u_r = u_z = 0,\ u_\theta = u_\theta\ (r,z,t) = u\,(r,z)\ e^{ipt} \\ U_r = U_z = 0,\ U_\theta = U_\theta\ (r,z,t) = U\,(r,z)\ e^{ipt} \end{aligned} \tag{1}$$

The equations of motion for a poroelastic solid in presence of dissipation[6] are

$$N\,\Delta^2\,\bar{u} + \text{grad}\,[(A+N)\,e + Q\epsilon] = \frac{\partial^2}{\partial t^2}\,(\rho_{11}\,\bar{u} + \rho_{12}\,\bar{U}) + b\,\frac{\partial}{\partial t}\,(\bar{u} - \bar{U}) \tag{2}$$

$$\text{grad}\ (Qe + R\epsilon) = \frac{\partial^2}{\partial t^2}(\rho_{12}\,\bar{u} + \rho_{22}\,\bar{U}) - b\,\frac{\partial}{\partial t}\,(\bar{u} - \bar{U})$$

where A, N, Q, R are elastic constants in Biot's theory. $\bar{u}, \bar{U}$ are displacement vectors for solid and fluid. The 'ρ's are the mass-coefficients such that the sums $\rho_{11} + \rho_{12}$ and $\rho_{12} + \rho_{22}$ represent mass of solid and fluid per unit volume of bulk material. The coefficient ρ_{12} is a mass-coupling parameter between fluid and solid phases. e and ϵ are average dilatation of solid and fluid which are

$$\begin{aligned} e &= \frac{1}{r}\,\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\ \epsilon &= \frac{1}{r}\,\frac{\partial(rU_r)}{\partial r} + \frac{1}{r}\,\frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z} \end{aligned} \tag{3}$$

Using (1), (3) gives $e = \epsilon = 0$ and excess pore pressure (s) developed in the system is

$$s = Qe + R\epsilon = 0$$

which is obvious because the waves considered here are shear waves.

The non-vanishing stress components interms of displacements are

$$\sigma_{r\theta} = Nr \frac{\partial}{\delta r}\left(\frac{u_\theta}{r}\right)$$

$$\sigma_{z\theta} = N \frac{\partial u_\theta}{\partial z} \tag{4}$$

Using (1) and (2), after simplification one obtains

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r^2} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} + \beta^2\right) u = 0 \tag{5}$$

with

$$\beta^2 = \frac{p^2}{N} \cdot \frac{\tau_{11}\,\tau_{22} - \tau^2_{12}}{\tau_{22}} \tag{6}$$

and

$$\tau_{11} = \rho_{11} - ib/p, \quad \tau_{12} = \rho_{12} + ib/p, \quad \tau_{22} = \rho_{22} - ib/p$$

If the viscous forces are not included in the equation of motion which is equivalent to putting $b = 0$, then $\tau_{ij} = \rho_{ij}$. Hence the theory including viscous losses is formally identical to the purely elastic theory if the densities are regarded as complex numbers.

Let the solution of (5) be

$$u\,(r, z) = RZ \tag{7}$$

where R is a function of r and Z is a function of z only.

Substitution of (7) into equation (5) gives

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\,\frac{dR}{dr} + \left(\beta^2 - \frac{1}{r^2}\right) R\right] = -\frac{d^2Z}{dz^2} = n^2$$

n being a constant.

The solution of it is

$$R = C_1\,J_1(qr) + C_2\,Y_1(qr)$$

$$Z = C_3 \cos n z + C_4 \sin n z \tag{8}$$

where

$$q^2 = \beta^2 - n^2 \tag{9}$$

and C_1, C_2, C_3, C_4 are arbitrary constants; $J_1\,(qr)$, $Y_1\,(qr)$ are Bessel functions of first and second kind of order one with argument qr.

The boundary conditions are ;

when

$$\left.\begin{array}{lll} (a)\ z = 0 \text{ and } l_1 & \sigma_{\theta z} = 0 \\ (b)\ r = a_1 \text{ and } a_2 & \sigma_{r\theta} = 0 \\ (c)\ z = 0, l_1; a_1, a_2 & s = 0 \end{array}\right\} \tag{10}$$

(10·c) is automatically satisfied.

$$\sigma_{\theta z} = 0 \quad \text{at } z = 0, \; l_1 \text{ gives}$$
$$C_4 = 0 \quad , n = m\pi/l_1 \tag{11}$$

where m is an integer $m = 0, \pm 1, \pm 2 \ldots\ldots$

Thus (8), (7), (1) with (11), gives

$$u_\theta = \{ A_1 J_1 (qr) + A_2 Y_1 (qr) \} \cos n z \, e^{ipt} \tag{12}$$

A_1 and A_2 being constants.

Substituting (12) in $\sigma_{r\theta}$ of (4) gives

$$\sigma_{r\theta} = N \cos n z \{ A_1 q J_2 (qr) + A_2 q Y_2 (qr) \} e^{ipt}$$

From boundary conditions (10 b) at $r = a_1$ and a_2, we have

$$A_1 q J_2 (q a_1) + A_2 q Y_2 (q a_1) = 0$$
$$A_1 q J_2 (q a_2) + A_2 q Y_2 (q a_2) = 0$$

The frequency follows from the conditions for a non-trivial solution to be

$$q^2 [J_2 (q a_1) Y_2 (q a_2) - J_2 (q a_2) Y_2 (q a_1)] = 0 \tag{13}$$

Setting

$$\delta = q a_1, \; \beta = a_2/a_1 \tag{14}$$

equation (13) can be re-written as

$$q^2 [J_2 (\delta) Y_2 (\delta\beta) - J_2 (\beta\delta) Y_2 (\delta)] = 0 \tag{15}$$

From (6), (9), (11) and (14) it follows that

$$p^2 = \frac{N \tau_{22} \left(\dfrac{\delta^2}{a_1^{\,2}} + \dfrac{m^2 \pi^2}{l_1^{\,2}} \right)}{\tau_{11} \tau_{22} - \tau_{12}^{\,2}}$$

which can be put in non-dimensional form as

$$(\sigma_{11} \sigma_{22} - \sigma_{12}^2) (a_1 p_1/c_0)^3 - b_1 (a_1 p_1/c_0)^2 + \sigma_{22} \{ \delta^2 + m^2 \pi^2 (a_1/l_1)^2 \} (a_1 p_1/c_0) -$$
$$- b_1 \{ \delta^2 + m^2 \pi^2 (a_1/l_1)^2 \} = 0 \tag{16}$$

where $p_1 = - ip$, $(a_1 p_1/c_0)$ is a non-dimensional frequency σ_{11}, σ_{12}, σ_{22} and b_1 are respectively non dimensional masses and dissipation coefficient defined by

$$\sigma_{11} = \rho_{11}/\rho, \; \sigma_{12} = \sigma_{12}/\rho, \; \sigma_{22} = \sigma_{22}/\rho, \; b_1 = a_1 b/\rho c_0$$

with

$$\rho = \rho_{11} + 2 \rho_{12} + \rho_{22}, \; c_0^{\,2} = N/\rho$$

Neglecting fluid affects the smallest value of p is

$$p = \frac{m\pi}{4} \left(\frac{N}{\rho_{11}} \right)^{\frac{1}{2}}$$

where $\rho_{11} = \rho$, in case of elasticity given by Stanisic and Osburn[5].

The numerical values of first and second roots of δ of vibration for various shape ratios $\beta = a_2/a_1$ for hollow cylinder were given by stanisic and Osburn[5]. The parameters of interest studied here are

$\beta = a_2/a_1$	δ_1	δ_2
(1) 0·0	5·136	8·420
(2) 0·5	6·814	12·855

$\beta = 0$ corresponds to solid cylinder of finite length. Frequency is calculated numerically for solid cylinder and hollow cylinder (outer radius is twice inner radius) for different masses, different values of a_1/l_1 and for various values of dissipation parameter. These are

	σ_{11}	σ_{22}	σ_{12}
(*i*)	0·50	0·50	0·00
(*ii*)	0·65	0·65	—0·15
(*iii*)	0·92	0·08	0·00

$$b_1 = 0{\cdot}01,\ 0{\cdot}1,\ 1,\ 10$$

$$a_1/l_1 = 1,\ 3/4,\ 1/2,\ 1/4$$

Of these first two materials are given by Biot[6] and third one corresponds to bone element.

DISCUSSION

In Tables 1, 2 and 3 the complex frequency of vibration in a solid cylinder of three different materials are given for various values of b_1 and ratio of radius to length of the cylinder. Frequency is found to decrease as the density of solid medium increases in the absence of mass-coupling effect and again increases when mass-coupling effect comes into picture, as given by Bout[6]. As the ratio of radius to length decreases the frequency also decreases.

TABLE 1

COMPLEX FREQUENCY OF VIBRATION IN A SOLID CYLINDER OF THREE DIFFERENT MATERIALS $\sigma_{11} = 0{\cdot}50$, $\sigma_{12} = 0{\cdot}00$ AND $\sigma_{22} = 0{\cdot}50$.

b_1	a_1/l_1	Solid cylinder	
		1st mode	2nd mode
0·01	1	0·030000 ± 8·732195 *i*	0·030001 ± 12·705885 *i*
	3/4	0·029998 ± 8·222852 *i*	0·029997 ± 12·361375 *i*
	1/2	0·029997 ± 7·838801 *i*	0·030000 ± 12·109298 *i*
	1/4	0·029999 ± 7·838801 *i*	0·030001 ± 11·955501 *i*
0·10	1	0·299839 ± 8·733886 *i*	0·299922 ± 12·707050 *i*
	3/4	0·299819 ± 8·224646 *i*	0·299916 ± 12·362573 *i*
	1/2	0·299805 ± 7·840683 *i*	0·299916 ± 12·110521 *i*
	1/4	0·299793 ± 7·601000 *i*	0·299915 ± 11·956738 *i*
1·00	1	2·882300 ± 8·838297 *i*	2·936232 ± 12·798973 *i*
	3/4	2·871111 ± 8·329221 *i*	2·933143 ± 12·455848 *i*
	1/2	2·861681 ± 7·944854 *i*	2·930740 ± 12·204778 *i*
	1/4	2·855309 ± 7·704628 *i*	2·929224 ± 12·051596 *i*

TABLE 2

COMPLEX FREQUENCY OF VIBRATION IN A SOLID CYLINDER OF THREE DIFFERENT MATERIALS $\sigma_{11} = 0{\cdot}65$, $\sigma_{12} = -0{\cdot}15$ AND $\sigma_{22} = 0{\cdot}65$.

b_1	a_1/l_1	Solid cylinder	
		1st mode	2nd mode
0·01	1	0·020193 ± 7·871091 *i*	0·020192 ± 11·452929 *i*
	3/4	0·020191 ± 7·411976 *i*	0·020192 ± 11·142391 *i*
	1/2	0·020193 ± 7·065797 *i*	0·020193 ± 10·915172 *i*
	1/4	0·020192 ± 6·849696 *i*	0·020193 ± 10·776540 *i*
0·10	1	0·201847 ± 7·872430 *i*	0·201884 ± 11·453851 *i*
	3/4	0·201841 ± 7·413399 *i*	0·201886 ± 11·143338 *i*
	1/2	0·201830 ± 7·067288 *i*	0·201884 ± 10·916140 *i*
	1/4	0·201823 ± 6·851234 *i*	0·201880 ± 10·777520 *i*
1·00	1	1·956234 ± 7·978298 *i*	1·986384 ± 11·535796 *i*
	3/4	1·949725 ± 7·522820 *i*	1·984712 ± 11·227036 *i*
	1/2	1·944174 ± 7·179403 *i*	1·983412 ± 11·001158 *i*
	1/4	1·940386 ± 6·965019 *i*	1·982585 ± 10·863360 *i*
10·00	1	x x	15·604858 ± 9·025577 *i*
	3/4	x x	15·545815 ± 8·478144 *i*
	1/2	x x	15·501984 ± 8·060760 *i*
	1/4	x x	15·474976 ± 7·798048 *i*

TABLE 3

COMPLEX FREQUENCY OF VIBRATION IN A SOLID CYLINDER OF THREE DIFFERENT MATERIALS $\sigma_{11} = 0\cdot92$, $\sigma_{12} = 0\cdot00$ AND $\sigma_{22}=0\cdot08$.

b_1	a_1/l_1	Solid cylinder 1st mode	Solid cylinder 2nd mode
0·01	1	0·133226 ± 6·432467 *i*	0·133254 ± 9·358676 *i*
	3/4	0·133222 ± 6·057416 *i*	0·133258 ± 9·104970 *i*
	1/2	0·133215 ± 5·774629 *i*	0·133251 ± 8·919335 *i*
	1/4	0·133211 ± 5·598104 *i*	0·133254 ± 8·806076 *i*
0·10	1	1·288676 ± 6·539682 *i*	1·310072 ± 9·438388 *i*
	3/4	1·283997 ± 6·169276 *i*	1·308896 ± 9·186554 *i*
	1/2	1·279988 ± 5·890190 *i*	1·307980 ± 9·002343 *i*
	1/4	1·277244 ± 5·716059 *i*	1·307398 ± 8·889976 *i*
1·00	1	9·119736 ± 6·154394 *i*	9·762878 ± 10·132904 *i*
	3/4	9·025879 ± 5·539562 *i*	9·712654 ± 9·820914 *i*
	1/2	8·953155 ± 5·043387 *i*	9·675331 ± 9·590049 *i*
	1/4	8·906868 ± 4·715093 *i*	9·652176 ± 9·447810 *i*

In Tables 4, 5, and 6 the complex frequency of a hollow cylinder whose thickness is equal to the inner radius are given for three different materials; (1) for various values of b_1 and (2) ratio of radius to length. The same conclusions as in above case are valid. The cross in the table indicate that vibrations do not exist for those values of parameters.

TABLE 4

COMPLEX FREQUENCY OF VIBRATION IN A HOLLOW CYLINDER OF THREE DIFFERENT MATERIALS $\sigma_{11} = 0\cdot50$ $\sigma_{12} = 0\cdot00$ AND $\sigma_{22}=0\cdot50$.

b_1	a_1/l_1	Hollow cylinder 1st mode	Hollow cylinder 2nd mode
0·01	1	0·030000 ± 10·611349 *i*	0·030005 ± 18·714737 *i*
	3/4	0·029999 ± 10·196314 *i*	0·029996 ± 18·482574 *i*
	1/2	0·029999 ± 9·889202 *i*	0·030005 ± 18·314941 *i*
	1/4	0·029997 ± 9·700269 *i*	0·030005 ± 18·213623 *i*
0·10	1	0·299894 ± 10·612743 *i*	0·2999962 ± 18·715530 *i*
	3/4	0·299888 ± 10·197764 *i*	0·299962 ± 18·483368 *i*
	1/2	0·299880 ± 9·890697 *i*	0·299971 ± 18·315750 *i*
	1/4	0·299871 ± 9·701793 *i*	0·299962 ± 18·214432 *i*
1·00	1	2·913609 ± 10·712559 *i*	2·968226 ± 18·786331 *i*
	3/4	2·907797 ± 10·298931 *i*	2·967484 ± 18·554871 *i*
	1/2	2·903146 ± 9·992779 *i*	2·966921 ± 18·387741 *i*
	1/4	2·900114 ± 9·804385 *i*	2·966583 ± 18·286743 *i*

TABLE 5

COMPLEX FREQUENCY OF VIBRATION IN A HOLLOW CYLINDER OF THREE DIFFERENT MATERIALS $\sigma_{11} = 0\cdot65$, $\sigma_{12} = 0\cdot15$ AND $\sigma_{22} = 0\cdot65$.

b_1	a_1/l_1	Hollow cylinder 1st mode	Hollow cylinder 2nd mode
0·01	1	0·020189 ± 9·564938 *i*	0·020192 ± 16·869247 *i*
	3/4	0·020191 ± 9·190831 *i*	0·020196 ± 16·659958 *i*
	1/2	0·020196 ± 8·914004 *i*	0·020192 ± 16·508865 *i*
	1/4	0·020194 ± 8·743702 *i*	0·020189 ± 16·417562 *i*
0·10	1	0·201872 ± 9·566042 *i*	0·201907 ± 16·869858 *i*
	3/4	0·201869 ± 9·191979 *i*	0·201908 ± 16·660599 *i*
	1/2	0·201860 ± 8·915188 *i*	0·201905 ± 16·509506 *i*
	1/4	0·201862 ± 8·744908 *i*	0·201902 ± 16·418167 *i*
1·00	1	1·973995 ± 9·659598 *i*	2·003256 ± 16·929092 *i*
	3/4	1·970754 ± 9·288112 *i*	2·002863 ± 16·720474 *i*
	1/2	1·968139 ± 9·013284 *i*	2·002593 ± 16·569870 *i*
	1/4	1·966423 ± 8·844232 *i*	2·002405 ± 16·478836 *i*
10·00	1	15·229904 ± 5·083576 *i*	16·493988 ± 16·614044 *i*
	3/4	15·150841 ± 3·946270 *i*	16·464005 ± 16·356766 *i*
	1/2	15·091248 ± 2·852038 *i*	16·442154 ± 16·169952 *i*
	1/4	15·054123 ± 1·904152 *i*	16·428879 ± 16·056625 *i*

TABLE 6

COMPLEX FREQUENCY OF VIBRATION IN A HOLLOW CYLINDER OF THREE DIFFERENT MATERIALS $\sigma_{11} = 0 \cdot 92$, $\sigma_{12} = -0 \cdot 00$ AND $\sigma_{22} = 0 \cdot 08$

b_1	a_1/l_1	Hollow cylinder 1st mode	Hollow cylinder 2nd mode
0·01	1	$0 \cdot 133247 \pm 7 \cdot 816233\, i$	$0 \cdot 133270 \pm 13 \cdot 783891\, i$
	3/4	$0 \cdot 133245 \pm 7 \cdot 510602\, i$	$0 \cdot 133271 \pm 13 \cdot 612909\, i$
	1/2	$0 \cdot 133241 \pm 7 \cdot 284450\, i$	$0 \cdot 133267 \pm 13 \cdot 489453\, i$
	1/4	$0 \cdot 133237 \pm 7 \cdot 145324\, i$	$0 \cdot 133270 \pm 13 \cdot 414834\, i$
0·10	1	$1 \cdot 301335 \pm 7 \cdot 908702\, i$	$1 \cdot 321828 \pm 13.840307\, i$
	3/4	$1 \cdot 299037 \pm 7 \cdot 606027\, i$	$1 \cdot 321567 \pm 13 \cdot 669983\, i$
	1/2	$1 \cdot 297117 \pm 7 \cdot 382168\, i$	$1 \cdot 321366 \pm 13 \cdot 547009\, i$
	1/4	$1 \cdot 295958 \pm 7 \cdot 244496\, i$	$1 \cdot 321250 \pm 13 \cdot 472686\, i$
1·00	1	$9 \cdot 442177 \pm 8 \cdot 161114\, i$	$10 \cdot 506645 \pm 15 \cdot 155230\, i$
	3/4	$9 \cdot 374138 \pm 7 \cdot 743797\, i$	$10 \cdot 481934 \pm 14 \cdot 970843\, i$
	1/2	$9 \cdot 322525 \pm 7 \cdot 426783\, i$	$10 \cdot 463905 \pm 14 \cdot 837370\, i$
	1/4	$9 \cdot 290314 \pm 7 \cdot 228034\, i$	$10 \cdot 452942 \pm 14 \cdot 756567\, i$

Numerical work was carried on when the non-dimensional parameter b_1 is equal to 10. It is observed that the frequency equation has real roots and hence no vibratory motion exists for the materials.

$$(i) \quad \sigma_{11} = 0 \cdot 50 \qquad \sigma_{22} = 0 \cdot 50 \qquad \sigma_{12} = 0 \cdot 00$$

$$(ii) \quad \sigma_{11} = 0 \cdot 92 \qquad \sigma_{22} = 0 \cdot 08 \qquad \sigma_{12} = 0 \cdot 00$$

Exceptionally in the case of hollow cylinder in the second mode, the complex frequency for material cited (*i*) above is given by

$$24.455490 \pm 13.711852\, i$$

$$24.418304 \pm 13.300437\, i$$

$$24.391281 \pm 12.997577\, i$$

$$24.374847 \pm 12.812004\, i$$

corresponding to value of $1 \cdot 00, 0 \cdot 75, 0 \cdot 50, 0 \cdot 25$ of the ratio of outer radius to the length respectively.

ACKNOWLEDGEMENT

We thank college authorities for providing financial assistance in carrying out the numerical work involved.

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