LARGE AMPLITUDE FREE VIBRATIONS OF PLATES RESTING ON A PASTERNAK-TYPE ELASTIC FOUNDATION

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In this paper following Berger's approximate plate theory for large deflections, large amplitude free vibrations of Rectangular plate, isosceles right-angled triangular plate, Equilateral triangular plate and circular plate resting on a Pasternak-type elastic foundation have been discussed.

Large deflection of flat isotropic plates has been investigated by the use of the approximate method offered by Berger[1]. This approximate method is based on neglecting the second invariant of middle surface strains in the expression for the total potential energy of the system. Iwinski and Nowinski[2] extended the method to orthotropic plate problems. Nowinski[3] has also solved same boundary value problems associated with circular and rectangular plates undergoing large deflection. Nash and Modeer[4] found the large amplitude free vibrations of rectangular and circular plates applying the technique shown by Berger Large amplitude free vibrations of isosceles right-angled triangular plates and elliptic plates have been. investigated by Banerjee[5,6]. Banerjee[7] found out the non-linear free and forced vibrations of different orthotropic plates.

The object of this paper is to investigate the large amplitude free vibrations of different plates resting on a Pasternak-type of elastic foundation by Berger Method. Numerical results are also presented in the form of graphs for the case of a simply supported rectangular plate.

FORMULATION OF THE PROBLEMS

The strain energy due to bending and stretching of the middle surface in a thin plate undergoing large deflections is given by Nash and Modeer[4].

$$V_1 = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e^2 - 2(1-\nu)\left\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right\} \right] dx\, dy \qquad (1)$$

where

w = deflection of the plate normal to the middle plane,

∇^2 = Laplacian operator,

D = flexural rigidity of the plate $= \dfrac{E h^3}{12(1-\nu^2)}$,

E = Young's modulus,

h = thickness of the plate,

ν = Poisson's ratio,

e = first invariant of the middle surface strains $= \epsilon_x + \epsilon_y$,

e_2 = second invariant of the middle surface strains

$= \epsilon_x \epsilon_y - \frac{1}{4} \gamma^2_{xy}$,

$$\epsilon_x = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 ,$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 ,$$

$$\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\ \frac{\partial w}{\partial y} ,$$

u, v = displacements along x— and y— axes respectively.

The foundation reaction p of the Pasternak elastic model is given by Kerr[8]

$$p = Kw - G\,\nabla^2 w \ , \tag{2}$$

where K is the foundation constant and G (a second foundation constant) is the shear modulus.

The potential energy V_2 of the distributed foundation reaction $p\,(x, y)$ as given by (2) is

$$V_2 = -\tfrac{1}{2} \iint \left[Kw^2 - G\left\{ \left(\frac{\partial w^2}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} \right] dx\, dy \tag{3}$$

Adding (1) and (3), the total potential energy $V_1 - V_2$ of the system takes the form[9]

$$V = \frac{D}{2} \iint \left[(\nabla^2 w)^2 + \frac{12}{h^2}\, e^2 - 2\,(1-v) \left\{ \frac{12}{h^2}\, e_2 + \frac{\partial^2 w}{\partial x^2}\ \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] + \frac{Kw^2}{D} - \frac{G}{D}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] \right] dx\, dy \tag{4}$$

The kinetic energy of the plate is

$$T = \frac{\rho h}{2} \iint \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right] dx\, dy , \tag{5}$$

where ρ denotes the density of the plate material.

Neglecting e_2 and applying Hamilton's principle and Euler differential equations of the variational problem, we finally obtain the following approximate differential equation for w in the absence of inertia effects in the plane of the plate[3] :

$$\nabla^4 w - \left[\alpha^2 f(t) - \frac{G}{D} \right] \nabla^2 w + \frac{Kw}{D} + \frac{12}{h^2 C\rho^2}\ \frac{\partial^2 w}{\partial t^2} = 0 , \tag{6}$$

where

$$C\rho^{-2} = \frac{\rho h^3}{12D}$$

and

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \tfrac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] = \frac{\alpha^2 h^2}{12} f(t) , \tag{7}$$

where α is a real normalised constan of integration.

SOLUTION OF THE PROBLEMS

Rectangular plate with all the edges simply supported

Let us consider the free vibrations of a flat rectangular plate with sides of lengths $2a$ and $2b$ in the x and y directions respectively and with the centre as the origin of co-ordinates. The deflections are considered to have the order of magnitude of the plate thickness.

The boundary conditions for simply supported edges are

$$\left.\begin{array}{l} u = w = \dfrac{\partial^2 w}{\partial x^2} = 0 \quad \text{on} \quad x = \pm a \\ v = w = \dfrac{\partial^2 w}{\partial y^2} = 0 \quad \text{on} \quad y = \pm b \end{array}\right\} \tag{8}$$

Let us assume u, v, w, in the following forms satisfying conditions (8)

$$u(x, y, t) = \sum_{s=1}^{\infty} g_s(y) \sin \alpha_s x \cdot H(t) \quad , \tag{9}$$

$$v(x, y, t) = \sum_{s=0}^{\infty} l_s(y) \cos \alpha_s x \cdot Q(t) \quad , \tag{10}$$

$$w(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \beta_m x \cdot \cos \gamma_n y \cdot F(t) \quad , \tag{11}$$

in which

$$\alpha_s = \frac{s\pi}{a} \quad ,$$

$$\beta_m = \frac{(2m+1)\pi}{2a} \quad , \gamma_n = \frac{(2n+1)\pi}{2b}$$

To determine the fundamental mode of vibration we put $m = n = 0$ in (11). Considering (9), (10), (11) and (7) and taking

$$F^2(t) = Q(t) = H(t) = f(t) \tag{12}$$

we obtain

$$\alpha^2 = \frac{3\pi^2 A^2_{00}}{8h^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \tag{13}$$

Inserting (11), (12) and (13) in (6) with $m = n = 0$, we finally get

$$\ddot{F} + \gamma F + \delta F^3 = 0 \quad , \tag{14}$$

where

$$\gamma = \frac{h^2 C\rho^2}{12}\left[\frac{\pi^2}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left\{\frac{\pi^2}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{G}{D}\right\} + \frac{K}{D}\right] \quad ,$$

$$\delta = \frac{\pi^4 A^2_{00} C\rho^2}{128}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2$$

which is to be solved subject to the initial conditions

$$F(0) = 1 \quad , \quad \dot{F}(0) = 0$$

Hence the solution of (14) is determined as

$$F(t) = cn\,(\omega^* t \ , \ \theta) \quad , \tag{15}$$

where

$$\omega^{*2} = \gamma + \delta \text{ and } \theta^2 = \frac{\delta}{2(\gamma+\delta)} .$$

Here ω^* and θ are positive constants for some particular values of K and G and cn is Jacobi's elliptic function.

The non-linear time period T^* of $cn\ (\omega^*\ t, \theta)$ is given by $T^* = \frac{4\Theta}{\omega^*}$, (16)

where Θ is the complete elliptic integral of the first kind.

The usual linear time period T is put as

$$T = \frac{2\pi}{\omega} ,$$

where ω is determined from the equation

$$\nabla^4 w + \frac{G}{D} \nabla^2 w + \frac{K}{D} w + \frac{12}{h^2 C_\rho^2} \frac{\partial^2 w}{\partial t^2} = 0 \qquad (17)$$

with

$$w = A_{00} \cos \beta_0 x \cdot \cos \gamma_0 y \cdot \cos \omega t.$$

Hence

$$\frac{T^*}{T} = \frac{2\Theta}{\pi} \cdot \frac{\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 + \frac{4}{\pi^4 D}\left\{4K - \pi^2 G\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}\right]^{\frac{1}{2}}}{\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\left(1+\frac{3}{2}\beta^2\right) + \frac{4}{\pi^4 D}\left\{4K - \pi^2 G\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}\right]^{\frac{1}{2}}} , \qquad (18)$$

where

$$\beta^2 = A^2_{00}/h^2$$

When K and G become zero in the limit, then equation (18) takes the form

$$\frac{T^*}{T} = \frac{2\Theta}{\pi}\left[1 + \frac{3}{2}\beta^2\right]^{-\frac{1}{2}} \qquad (19)$$

as investigated by Nash and Modeer[4] without any elastic foundation.

Simply supported isosceles right-angled triangular plate

Let us now consider that the edges of the flat plate resting on a pasternak-type of elastic foundation be $x = y = 0$ and $x + y = a$.

For such a plate the boundary conditions are

$$u = w = \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{at} \quad x = 0 ,$$

$$v = w = \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{at} \quad y = 0 ,$$

$$w = \frac{\partial^2 w}{\partial \eta^2} = 0 \qquad \text{at } x + y = a ,$$

where

$$\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

Compatible with the boundary conditions the suitable expressions for u, v, w can be taken as

$$u = \sum_{n=1,3,\ldots\ldots}^{\infty} B_n \sin \alpha_n x \left[\cos \alpha_n y + \sin \alpha_n x - \frac{n\pi}{4} \right] H(t),$$

$$v = \sum_{n=1,3,\ldots\ldots}^{\infty} B_n \sin \alpha_n y \left[\cos \alpha_n x - \sin \alpha_n y + \frac{n\pi}{4} \right] Q(t),$$

$$w = \sum_{m=1,3,\ldots\ldots}^{\infty} A_m \left[\sin 2\alpha_m x . \sin \alpha_m y + \sin 2\alpha_m y . \sin \alpha_m x \right] \times F(t),$$

$$\alpha_i = \frac{i\pi}{a}, \quad i = n, m .$$

Substituting the expressions of u, v and w in (7) and using (12), integration over the area of the plate yields after necessary simplification

$$\alpha^2 = 15 \left(\frac{A_1 \pi}{a h} \right)^2 \tag{20}$$

and the governing equation is given by

$$\ddot{F} + \gamma F + \delta F^3 = 0, \tag{21}$$

where

$$\gamma = \left[\frac{25\pi^4}{a^4} - \frac{5\pi^2 G}{a^2 D} + \frac{K}{D} \right] \frac{h^2 C_\rho^2}{12},$$

$$\delta = \frac{25\pi^4 A_1^2 C_\rho^2}{4a^4} .$$

Equation (21) can be solved as in the case of rectangular plate.

The ratio between the non-linear time period and the linear time period is

$$\frac{T^*}{T} = \frac{2\Theta}{\pi} \frac{\left[\frac{25}{a^4} + \frac{1}{\pi^4 D} \left(K - \frac{5\pi^2 G}{a^2} \right) \right]^{\frac{1}{2}}}{\left[\frac{25}{a^4} \left(1 + 3\beta^2 \right) + \frac{1}{\pi^4 D} \left(K - \frac{5\pi^2 G}{a^2} \right) \right]^{\frac{1}{2}}}, \tag{22}$$

where

$$\beta^2 = A_1^2 / h^2$$

leading to

$$\frac{T^*}{T} = \frac{2\Theta}{\pi} \left[1 + 3\beta^2 \right]^{-\frac{1}{2}} \tag{23}$$

in absence of elastic foundation as offered by Banerjee[5].

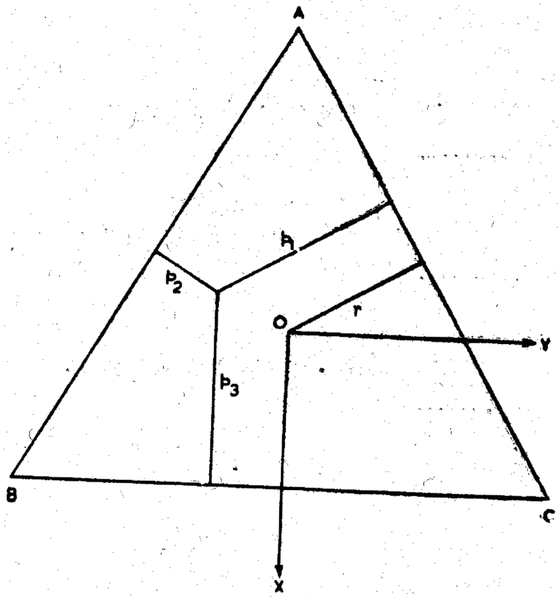

Fig. 1—Equilateral triangular flat plate.

Simply Supported Equilateral Triangular Plate

Trilinear co-ordinates : Let ABC be an equilateral triangle of side a. The centroid O in the undeflected middle surface is taken as the origin of co-ordinates. The axes OX and OY are taken perpendicular and parallel to BC respectively. If p_1, p_2, p_3 be three perpendiculars from a point $p\ (x, y)$ within the triangle on AC, AB and BC respectively, r, the radius of the inscribed circle, then

$$p_1 = r + \frac{x}{2} - \frac{\sqrt{3}}{2}\, y \ ,$$

$$p_2 = r + \frac{x}{2} + \frac{\sqrt{3}}{2}\, y \ ,$$

$$p_3 = r - x \ ,$$

and

$$p_1 + p_2 + p_3 = 3r = \frac{\sqrt{3}}{2}\, a = \lambda_a \ (say).$$

Let us take an equilateral triangular flat plate of side a resting on a Pasternak-type of elastic foundation.

In this case, we have the boundary conditions :

$$w = \nabla^2 w = 0 \quad \text{on} \quad p_1 = p_2 = p_3 = 0 \, .$$

we assume,

$$u = v = 0 \quad \text{on} \quad p_1 = p_2 = p_3 = 0 \, .$$

The above conditions will be satisfied if the expressions for u, v and w are taken in the following forms :

$$u = \sum_{m=1}^{\infty} \sqrt{3}\, B_m \left[\sin \delta_m \left(p_2 + p_3 \right) + \sin \delta_m \left(p_1 + p_3 \right) \right] H(t) \, ,$$

$$v = \sum_{m=1}^{\infty} B_m \left[\sin \delta_m \left(p_1 + p_3 \right) - \sin \delta_m \left(p_2 + p_3 \right) \right] Q(t) \, ,$$

$$w = \sum_{n=1}^{\infty} A_n \left[\sin \delta_n p_1 + \sin \delta_n p_2 + \sin \delta_n p_3 \right] F(t) \, ,$$

where

$$\delta_i = \frac{2\, i\, \pi}{\lambda_a} \ , \ i = m\, ,\, n\, .$$

Proceeding in the similar manner as deduced in the previous article, we thus obtain for the fundamental mode of vibration

$$\alpha^2 = \frac{48\, \pi^2\, A_1^2}{a^2\, h^2} \quad . \tag{24}$$

To evaluate $F(t)$ we have the required equation

$$\ddot{F} + \gamma_* F + \delta\, F^3 = 0 \quad , \tag{25}$$

where

$$\gamma = \left[\frac{16\, \pi^4}{\lambda_a^4} - \frac{4\, \pi^2\, G}{\lambda_a^2\, D} + \frac{K}{D} \right] \frac{h^2\, C_\rho^2}{12} \quad ,$$

$$\delta = \frac{16\, \pi^4\, A_1^2\, C_\rho^2}{\lambda_a^2\, a^2} \quad .$$

Here ratio of T^* and T is given by

$$\frac{T^*}{T} = \frac{2\, \Theta}{\pi} \cdot \frac{\left[\frac{16}{\lambda_a^4} + \frac{1}{\pi^4 D} \left(K - \frac{4\, \pi^2\, G}{\lambda_a^2} \right) \right]^{\frac{1}{2}}}{\left[\frac{16}{\lambda_a^4} \left(1 + 9\, \beta^2 \right) + \frac{1}{\pi^4 D} \left(K - \frac{4\, \pi^2\, G}{\lambda_a^2} \right) \right]^{\frac{1}{2}}} \quad , \tag{26}$$

where

$$\beta^2 = \frac{A_1^2}{h^2} \quad .$$

In the absence of elastic foundation $K = G = 0$.

Hence

$$\frac{T^*}{T} = \frac{2\, \Theta}{\pi} \left[1 + 9\ \beta^2 \right]^{-\frac{1}{2}} . \tag{27}$$

Circular Plate

An approximate analysis of a flat circular plate having its boundary elastically restrained against rotation is offered here. Let the circular plate of radius R be axisymmetric and be placed on an elastic foundation of Pasternak-type.

The transformations of equations (6) and (7) into polar co-ordinates lead to

$$\nabla^4 w - \left[\alpha^2 F^2(t) - \frac{G}{D} \right] \nabla^2\ w + \frac{K}{D}\ w + \frac{12}{h^2\, C_\rho^2}\ \frac{\partial^2 w}{\partial t^2} = 0 \tag{28}$$

and

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 = \frac{\alpha^2 h^2}{12} f(t) , \tag{29}$$

where

$$\nabla^2 = \frac{d^2}{d r^2} + \frac{1}{r} \frac{d}{d r} ,$$

$$f(t) = F^2(t) .$$

Let us assume that

$$u = f(r) F^2(t) , \tag{30}$$

$$w = W(r) F(t) . \tag{31}$$

Combining (28) and (31) we have

$$F(t) \nabla^4 W - \left[\alpha^2 F^2(t) - \frac{G}{D} \right] F(t) \nabla^2 W + \frac{K}{D} F(t) W + \frac{12}{h^2 C_\rho^2} \frac{d^2 F}{d t^2} W = 0 \tag{32}$$

A solution is possible if

$$\frac{\nabla^4 W}{W} = \varphi^4 \text{ and } \frac{\nabla^2 W}{W} = - \varphi^2 \tag{33}$$

Hence,

$$W = A J_o(\varphi r) , \tag{34}$$

where J_o is the Bessel function of the first kind of order zero.

Considering equations (32) and (33), we now obtain

$$\frac{d^2 F}{d t^2} + \left(\varphi^4 - \frac{\varphi^2 G}{D} + \frac{K}{D} \right) \frac{h^2 C_\rho^2}{12} . F + \frac{h^2 C_\rho^2 \varphi^2 \alpha^2}{12} F^3 = 0 \tag{35}$$

The solution of this equation is put as

$$F(\mathrm{t}) = cn(\omega^* \mathrm{t}, \theta) ,$$

where

$$\omega^{*2} = \frac{h^2 C_\rho^2 \varphi^4}{12} \left[1 + \frac{\alpha^2}{\varphi^2} + \frac{K}{D \varphi^4} - \frac{G}{D \varphi^2} \right],$$

$$\theta^2 = \frac{1}{2\left[1 + \frac{\varphi^2}{\alpha^2} + \frac{K}{D \alpha^2 \varphi^2} - \frac{G}{D \alpha^2} \right]} .$$

From (29) and (30) we get

$$f(r) = \frac{\alpha^2 h^2 r}{24} - \frac{A^2 \varphi^2 r}{4} \left[J_1^2(\varphi^2 r) + J_o^2(\varphi r) - \frac{2 J_o(\varphi r). J_1(\varphi r)}{\varphi r} \right]. \tag{36}$$

For w to vanish on the boundary $r = R$, φ must be a root of

$$J_o(\varphi R) = 0 . \tag{37}$$

Again for $u = 0$ on $r = R$, we have from (36)

$$\frac{a^2 h^2}{6} = A^2 \varphi^2 \left[J_1^2 (\varphi R) \right] \tag{38}$$

Thus the ratio T^*/T is given by

$$\frac{T^*}{T} = \frac{2\,\Theta}{\pi} \cdot \frac{1}{\left[1 + \frac{6 A^2}{h^2} \cdot J_1^2 (\varphi R) + \frac{K}{D \varphi^4} - \frac{G}{D \varphi^2} \right]^{\frac{1}{2}}} . \tag{39}$$

When K and G become zero in the limit, we get the corresponding result

$$\frac{T^*}{T} = \frac{2\,\Theta}{\pi} \left[1 + \frac{6A^2}{h^2} \cdot J_1^2 (\varphi R) \right]^{-\frac{1}{2}} \tag{40}$$

as obtained by Nash and Modeer[4].

NUMERICAL RESULTS

Numerical results are offered graphically in Fig. 2 for simply supported rectangular plate in terms of the aspect ratio $\frac{b}{a} = 1$ to give rise an idea of the variation of the ratio T^*/T given by (18) with respect to different values of β having considered some particular values of non-dimensional foundation modulus $\lambda \left(= \frac{K a^4}{D} \right)$ and $\mu \left(= \frac{G a^2}{D} \right)$.

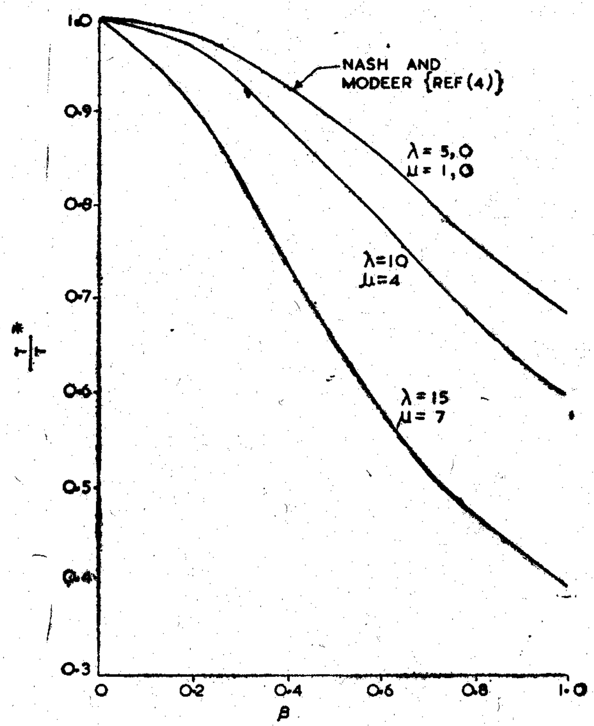

Fig. 2—Variation of the ratio of T^*/T with respect to different values of β.

It is apparent that the results of Nash and Modeer[4] for large amplitude free vibrations of rectangular plates without any elastic foundation coincides with that of the same plate with $b/a = 1$ for $\lambda = 5$, $\mu = 1$. These results are also plotted by way of comparison.

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