ON RADIATIVE BOUNDARY SHOCK WAVES

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In this paper, we have postulated in a radiative gas, the occurrence of a boundary shock wave, considered as a quick transition region in which the viscous effects are compressible and are confined to a thin layer adjacent to a surface. The generalised Rankine Hugoniot jump relations, Prandtl relation and the other properties of a shock wave are derived and a discussion about the existence of a boundary shock wave is made.

The possible occurrence of a boundary shock wave as considered by Martin[1] is generalised to the case of a radiative gas. Martin regarded a boundary shock wave as a thin region of viscous flow adjacent to a surface from which gas flows at a very high rate with large heat transfer. When the application of boundary conditions is considered in the flow of an inviscid bulk of vapour (or injected gas) at high Reynolds number the phenomena is possibly expected. Since the rapid decay of the viscous stress and heat conduction flux over a small distance would allow a rapid transition between an inviscid solution for the flow out of the surface and surface conditions, for example, heat conduction, the boundary shock wave is regarded as quick transition region belonging to the class of asymptotic phenomena discussed by Friedrichs[2]. It is like a boundary layer in the sense that the viscous effects are confined to the thin layer adjacent to the surface and is like a viscious shock wave in the sense that the flow is normal to the layer and the viscous effects are compressive. The occurrence of a boundary shock wave is also possible when a gas flows out of a porous wall if radiation is absorbed at the surface and is conducted back into the wall. Since extremely high temperature gases are involved in the phenomena, we have considered in this paper the boundary shock wave in a radiating gas. We have assumed the existence of a boundary shock wave and have derived across it the jumps in the flow variables. Prandtl relation and other properties pertaining to the present case are also obtained.

PLANE LAMINAR BOUNDARY SHOCK WAVE IN A RADIATIVE GAS WITH $\widetilde{Pr}=1$

We assume the flow to be closed enough to mechanical equilibrium. The geometrical structure of the surface is taken to be of such small detail that the velocity vector can be considered to be essentially one dimensional so that the flow equations in one dimensional steady flow in a non-accelerating co-ordinate system are

$$\frac{d}{dx}(\rho u) = 0 \tag{1}$$

$$\frac{d}{dx}(\rho u^2) = \frac{df}{dx} \tag{2}$$

$$\rho u \frac{d}{dx}\left(e + \frac{1}{2}u^2\right) = \frac{d}{dx}\left(-q + uf\right) \tag{3}$$

where x denotes the distance on the positive side of x-axis, f is the sum of the surface forces given by

$$f = -p - \frac{aT^4}{3} + \tau \tag{4}$$

$\frac{a\,T^4}{3}$ being the radiation pressure, e the internal energy per unit mass, τ the viscous stress, q the sum of the conduction and radiation flux q_c and q_r, $\widetilde{P}r$ the longitudinal Prandtl number, a the coefficient of heat radiation and other symbols have their usual meaning. The specific enthalpy h with radiation effect is given by

$$h = e + \frac{4}{3}\,\frac{aT^4}{\rho} + \frac{p}{\rho} \tag{5}$$

where $\frac{aT^4}{\rho}$ is the radiation energy per unit mass. The equation of state for a thermally and calorically perfect gas is

$$p = \rho RT \tag{6}$$

and the conduction and the radiation heat flux in the x-direction are given by

$$q_c = -\,k_c\,\frac{dT}{dx} \tag{7}$$

and

$$q_r = -\,k_r\,\frac{dT}{dx} \tag{8}$$

where k is the coefficient of thermal conductivity and k_r is the effective coefficient of heat conductivity by radiation.

The boundary conditions to be used along with equations (1), (2) and (3) are at $x = x_b = 0+$

$$u = u_b\,,\quad T = T_b\,,\quad q_c = q_{cb}\,,\quad q_r = q_{rb} \tag{9}$$

and as $x \to \infty$

$$\frac{du}{dx} \to 0\,,\ \frac{dT}{dx} \to 0 \tag{10}$$

where a quantity with suffix b indicates its value at the boundary in the gas ($x = 0+$) and with the suffix e its value in the gas outside the boundary shock ($x \to \infty$). We now define the conduction heat transfer coefficient and the radiation heat transfer coefficients as

$$C_{hc} \equiv \frac{-\,q_{cb}}{(\frac{1}{2})\,\rho_b\,u_b^3} = \frac{-\,\tau_b}{(\frac{1}{2})\,\rho_b\,u_b^2} \tag{11}$$

$$R_{hc} \equiv \frac{-\,q_{rb}}{(\frac{1}{2})\,\rho_b\,u_b^3}\ ;\ R'_{hc} \equiv \frac{-\,q_{re}}{(\frac{1}{2})\,\rho_e\,u_e^3} \tag{12}$$

CONDITIONS ACROSS A BOUNDARY SHOCK

Integrating equations (1), (2) and (3) and applying the boundary conditions (9) and (10), we have

$$\rho_b u_b = \rho_e u_e \tag{13}$$

$$\rho_b\,u_b^2 + p_b + \frac{a\,T_b^4}{3} - \tau_b = \rho_e\,u_e^2 + p_e + \frac{a\,T_e^4}{3} \tag{14}$$

$$\frac{1}{2}\,u_b^2 + \frac{\gamma}{\gamma-1}\,\frac{p_b}{\rho_b} + \frac{4a\,T_b^4}{3\,\rho_b} = \frac{1}{2}\,u_e^2 + \frac{\gamma}{\gamma-1}\,\frac{p_e}{\rho_e} + \frac{4a\,T_e^4}{3\rho_e} + \frac{Q}{\rho_b\,u_b} \tag{15}$$

where $Q = q_{re} - q_{rb}$ is the net transport of radiation energy from the shock front. It has been shown[1] that at every point in the boundary shock $q_c = u\tau$, ($\widetilde{P}r = 1$). On account of the assumption, $\widetilde{P}r = 1$, q_c and τ are eliminated from the equation (15). In addition, we have assumed here that the radiation flux is independent of τ.

As a consequence of equations (13) and (14), we obtain a relation analogous to Rankine-Hugoniot relation in the form

$$\frac{p_b}{\rho_b u_b} - \frac{p_e}{\rho_e u_e} = u_e - u_b + \frac{\tau_b}{\rho_b u_b} + \frac{a}{3}\left(\frac{T_e^4}{\rho_e u_e} - \frac{T_b^4}{\rho_b u_b}\right) \tag{16}$$

Multiplying equation (16) by $(u_e + u_b)$ and making use of (15), we obtain a relation analogous to the 'Hugoniot relation' as

$$\left(p_e - p_b\right)\left(\frac{1}{\rho_e} + \frac{1}{\rho_b}\right) + \frac{\tau_b}{\rho_b}\left(1 + \frac{\rho_b}{\rho_e}\right) + \frac{a}{3}\left(T_e^4 - T_b^4\right)\left(\frac{1}{\rho_e} + \frac{1}{\rho_b}\right) =$$

$$= \frac{2\gamma}{\gamma - 1}\left(\frac{p_e}{\rho_e} - \frac{p_b}{\rho_b}\right) + \frac{8a}{3}\left(\frac{T_e^4}{\rho_e} - \frac{T_b^4}{\rho_b}\right) + \frac{2Q}{\rho_b u_b} \tag{17}$$

This relation can be rearranged to obtain a form which, according to Liepmann and Roshko[4] is analogous to the Rankine-Hugoniot relation

$$\frac{\rho_b}{\rho_e} = \frac{1 + \dfrac{\gamma + 1}{\gamma - 1}\dfrac{p_e}{p_b} - \dfrac{\tau_b}{p_b} + \dfrac{a}{3p_b}\left(T_e^4 + 7T_b^4\right)}{\dfrac{\gamma + 1}{\gamma - 1} + \dfrac{p_e}{p_b} + \dfrac{\tau_b}{p_b} + \dfrac{a}{3p_b}\left(7T_e^4 + T_b^4\right) + \dfrac{2Q}{u_b p_b}} \tag{18}$$

Prandtl Relation

Denoting the sound speed and radiation sound speed respectively by

$$C = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}$$

and

$$C_r = \sqrt{\frac{\gamma a T^4}{3\rho}},$$

we can write the equations (16) and (15) as

$$u_b - u_e = \frac{C_e^2}{\gamma u_e} - \frac{C_b^2}{\gamma u_b} + \frac{\tau_b}{\rho_b u_b} + \frac{C_{re}^2}{\gamma u_e} - \frac{C_{rb}^2}{\gamma u_b} \tag{19}$$

and

$$\frac{u_b^2}{2} + \frac{C_b^2}{\gamma - 1} + \frac{4C_{rb}^2}{\gamma} + \frac{q_{rb}}{\rho_b u_b} = \frac{u_e^2}{2} + \frac{C_e^2}{\gamma - 1} + \frac{4C_{re}^2}{\gamma} +$$

$$+ \frac{q_{re}}{\rho_e u_e} = \frac{1}{2}\frac{\gamma + 1}{\gamma - 1}C^{*2} \equiv D \tag{20}$$

respectively, where

$$C^{*2} = \frac{(\gamma - 1)u_b^2 + 2C_b^2 + \dfrac{8(\gamma - 1)}{\gamma}C_{rb}^2 + \dfrac{2(\gamma - 1)}{\rho_b u_b}q_{rb}}{\gamma + 1} \tag{21}$$

From equation (20), we get

$$C_b^2 = (\gamma - 1)D - \frac{(\gamma - 1)}{2}u_b^2 - \frac{4(\gamma - 1)}{\gamma}C_{rb}^2 - \frac{(\gamma - 1)q_{rb}}{\rho_b u_b} \tag{22}$$

and

$$C_e^2 = (\gamma - 1)D - \frac{(\gamma - 1)}{2}u_e^2 - \frac{4(\gamma - 1)}{\gamma}C_{re}^2 - \frac{(\gamma - 1)q_{re}}{\rho_e u_e} \tag{23}$$

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Substituting equations (22) and (23) in (19), we get

$$u_b - u_e = C^{*2}\left(\frac{1}{u_e} - \frac{1}{u_b}\right) + \frac{2\gamma}{\gamma+1}\left(\frac{\tau_b}{\rho_b u_b}\right) - \frac{(6\gamma-8)}{\gamma(\gamma+1)}\left(\frac{C_{re}^2}{u_e} - \frac{C_{rb}^2}{u_b}\right) - \\ - \frac{2(\gamma-1)}{(\gamma+1)\rho_b u_b}\left(\frac{q_{re}}{u_e} - \frac{q_{rb}}{u_b}\right) . \tag{24}$$

which can be further simplified to give

$$u_b u_e = C^{*2} + \frac{2}{(\gamma+1)(\rho_e - \rho_b)}\left\{\gamma\tau_b - \frac{(3\gamma-4)}{\gamma}\left(\rho_e C_{re}^2 - \rho_b C_{rb}^2\right) - \\ - (\gamma-1)\left(\frac{q_{re}}{u_e} - \frac{q_{rb}}{u_b}\right)\right\} \tag{25}$$

This gives the generalised Prandtl relation for the present case. It can be easily seen that for a simple normal shock wave in a non-radiating gas, the equation (25) reduces to the well-known Prandtl relation

$$u_b u_e = C^{*2} \tag{26}$$

which determines that the flow through a normal shock wave must go from either supersonic to subsonic or vice-versa. For a boundary shock wave in a radiative gas, however, with $\tau_b \neq 0$, $C_{re}, C_{rb} \neq 0$, $q_{re}, q_{rb} \neq 0$ such a relation can not be imposed and both u_b and u_e may be subsonic. By equating the two expressions for $\left(\frac{C^*}{u_b}\right)^2$ obtained from the equations (20) and (25), we get on rearrangement

$$A_1 Z^2 - A_2 Z + A_3 = 0 \tag{27}$$

where

$$Z = \frac{\rho_b}{\rho_e}$$

$$A_1 = \left[1 + \frac{\gamma-1}{\gamma+1} R'_{hc} - \frac{(6\gamma-8)}{\gamma(\gamma+1) M_{re}^2}\right]$$

$$A_2 = \frac{2\gamma}{\gamma+1}\left[1 + \frac{1}{\gamma M_b^2} + \frac{1}{2} C_{hc} + \frac{1}{\gamma M_{rb}^2}\right]$$

$$A_3 = \frac{2\gamma}{\gamma+1}\left[\frac{\gamma-1}{2\gamma} + \frac{1}{\gamma M_b^2} + \frac{4(\gamma-1)}{\gamma^2 M_{rb}^2} - \frac{\gamma-1}{2\gamma} R_{hc}\right]$$

and

$$M_b = \frac{u_b}{C_b}$$

$$M_{rb} = \frac{u_b}{C_{rb}} \tag{28}$$

For given values of γ, M_b, M_{rb}, R_{hc} and C_{hc}, the ratio $\frac{\rho_b}{\rho_e}$ $(= Z)$ can be obtained. In a similar manner the pressure ratio and the temperature ratio may be obtained as

$$\frac{p_e}{p_b} = 1 + \gamma M_b^2\left(1 + \frac{1}{2} C_{hc} - \frac{\rho_b}{\rho_e}\right) + \frac{a}{3p_b}\left(T_b^4 - T_e^4\right) \tag{29}$$

and

$$\frac{T_e}{T_b} = 1 + \frac{\gamma - 1}{2} M_b^2 \left[1 - \left(\frac{\rho_b}{\rho_e} \right)^2 \right] + \frac{4(\gamma - 1)}{\gamma C_b^2} \left(C_{rb}^2 - C_{re}^2 \right) + + \frac{\gamma - 1}{C_b^2} \left(\frac{q_{rb}}{\rho_b u_b} - \frac{q_{re}}{\rho_e u_e} \right) . \tag{30}$$

The Mach number Me downstream of the boundary shock can be found by writing the equation (14) in the form

$$\gamma Me^2 = \left[\frac{\rho_b}{\rho_e} \left\{ 1 + \frac{1}{\gamma M_b^2} + \frac{1}{2} C_{hc} + \frac{a}{3 \rho_b u_b^2} \left(T_b^4 - T_e^4 \right) \right\} - 1 \right]^{-1} . \tag{31}$$

Me^2 can also be written as

$$Me^2 = \frac{u_e^2}{\gamma R T_e} \frac{\gamma R T_b}{u_b^2} M_b^2$$

into which the equations (13) and (31) may be substituted to give

$$Me^2 = \frac{\left(\frac{\rho_b}{\rho_e} \right)^2 M_b^2}{1 + \frac{\gamma - 1}{2} M_b^2 \left[1 - \left(\frac{\rho_b}{\rho_e} \right)^2 \right] + \frac{4'(\gamma - 1)}{\gamma C_b^2} \left(C_{rb}^2 - C_{re}^2 \right) + + \frac{\gamma - 1}{C_b^2} \left(\frac{q_{rb}}{\rho_b u_e} - \frac{q_{re}}{\rho_e u_e} \right)} . \tag{32}$$

The entropy production Δ_{iS} must be positive in order to ensure the physical conditions required by the equations derived so far and hence the conservation equations have to be supplemented by the second law of thermodynamics which can be written as

$$\Delta_{iS} = \Delta S - \Delta_{eS} = S_e - S_b - \frac{q_{cb}}{\rho_b u_b T_b} - \frac{q_{rb}}{\rho_b u_b T_b} \geqslant 0 \tag{33}$$

where Δ_{eS} is the change in entropy due to heat transfer and S is the specific entropy. Equation (33) can also be written as

$$\frac{\Delta_{iS}}{R} = \frac{S_e - S_b}{R} + \frac{1}{2} \gamma M_b^2 C_{hc} + \frac{1}{2} \gamma M_b^2 R_{hc} \geqslant 0 \tag{34}$$

For a perfect gas, the entropy is given by

$$\frac{\gamma - 1}{R} \left(S_e - S_b \right) = \log \left[\frac{p_e^*}{p_b^*} \left(\frac{\rho_b}{\rho_e} \right)^{\gamma} \right] \tag{35}$$

where we have taken S_b as the reference value and S_e as the value of interest and p^* is the sum of gas pressure and radiation pressure. Then the equations (33), (34) and (35) give us the required conditions as

$$\frac{\gamma - 1}{R} \Delta_{iS} = \log \left[\frac{p_e^*}{p_b^*} \left(\frac{\rho_b}{\rho_e} \right)^{\gamma} \right] + \frac{1}{2} \gamma (\gamma - 1) M_b^2 C_{hc} + + \frac{1}{2} \gamma (\gamma - 1) M_b^2 R_{hc} \geqslant 0 \tag{36}$$

In order that the Mach number Me, the density ratio, pressure and temperature ratios across a boundary shock wave be physically possible, the equation (27) must have a real positive root, the condition for which is

$$2 + \frac{2}{\gamma M_b^2} + \frac{2}{\gamma M_{rb}^2} + C_{hc} > 0 \tag{37}$$

It is obvious that for positive value of C_{hc} this condition is always satisfied. For negative value of C_{hc}, however, we must have

$$2 + \frac{2}{\gamma M_b^2} + \frac{2}{\gamma M_{rb}^2} > C_{hc} \tag{38}$$

Since boundary shock wave is a gas-dynamic discontinuity, its existence depends on the stability of the phenomena c.f. Hayes[5]. For positive C_{hc} the heat is conducted back through the out-flowing fluid, in the same direction as the heat conduction occurs in the physical situation existing in a simple shock wave. As such the cases with positive C_{hc} only are significant since they correspond to simple shock wave solutions known to exist in reality and be stable.

The physical existence of a boundary shock seems doubtful for negative values of C_{hc} even in those cases where C_{hc} satisfies the equation (37). The structure of a boundary shock wave in a radiative gas is also being studied and will be communicated shortly.

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