HYDROMAGNETIC FLOW BETWEEN TWO PARALLEL POROUS WALLS IN A ROTATING SYSTEM

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The effect of uniform transverse magnetic field is investigated on the flow formed, when a straight channel formed by two parallel porous walls through which liquid is flowing under a constant pressure gradient, is rotated about an axis perpendicular to the walls. The flow depends on the Taylor's number α, Pressure gradient P, the suction Reynolds number β and the Hartmann number M. When $\alpha \to \infty$ such that P is finite, thin boundary layers are formed in the vicinity of the porous walls and the effect of the magnetic field is to reduce the thickness of the boundary layers. A method for setting up an experiment to test the theoretical conclusions of the paper has been suggested.

The fundamental difficulty in solving the Navier-Stokes equations either exactly or approximately is the non-linearity introduced by the convection terms in the momentum equations. There exist, however, non-trivial problems in which the convection terms vanish and these provide the simple class of solutions of the equations of motion. One such flow has been considered recently by Vidyanidhi and Nigam[1] who have studied the secoudary flow when a straight channel formed by two parallel walls, through which liquid is flowing under a constant pressure gradient is rotated about an axis perpendicular to the walls. This problem was later extended by Vidyanidhi[2] in the frame-work of hydromagnetics.

Recently, the problem of fluid flow through porous ducts has because of its application to the cases of transpiration cooling, gaseous diffusion, etc., become a subject of study by numerous authors. The effect of suction in rectangular channels was studied by Berman[3] and later extended by Surya Prakasa Rao[4], Terrill and Shrestha[5] etc., in the frame-work of hydromagnetics, neglecting the induced magnetic field. Similar work due to Reddy and Jain[6], Vidyanidhi and Ramana Rao[7] has also been found in literature. The effects of uniform suction for the Karman problem of flow and other flows in rotating frame of reference have been discussed by numerous authors Stuart[8], Gupta[9], Debnath and Mukherjee[10], and Singh and Sathi[11]. Recently Vidyanidhi, Bala Prasad and Ramana Rao[12] have studied the effects of uniform suction and injection on the flow investigated by Vidyanidhi and Nigam[1]. In all these problems dealing with suction and injection, it is seen that, even if the convection terms in the momentum equation do not vanish, yet they did not introduce serious complications as they are linear in the unknown variables.

The object of the present paper is as such to extend our earlier problem[12] in the frame-work of hydromagnetics, neglecting the induced magnetic field, following the works of several authors[4—7]. It is of interest to examine the nature of the secondary flow which is set up due to the interaction between the pressure gradient and Coriolis forces. Further it affords a simple picture of the way in which the thickness of the boundary layers that arise in a rapidly rotating system is reduced as an effect of the magnetic field.

THE BASIC EQUATIONS AND SOLUTIONS

The basic equations which express the interactions between the fluid motions and the magnetic fields are Maxwell's equations and the Navier-Stokes equations. The steady motion of an incompressible conducting fluid in the presence of a magnetic field in a rotating frame of reference $0' X' Y' Z'$ is governed by the following equations (in rationalized MKS system of units).

Maxwell's equations :

$$\vec{\nabla}' \cdot \vec{B} = 0, \quad \vec{\nabla}' \times \vec{E} = 0, \quad \vec{\nabla}' \times \vec{H} = \vec{J}, \tag{1}$$

where

$$\vec{B} = \mu_e \vec{H}. \tag{2}$$

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Ohm's law for moving media can be written as

$$\vec{J} = \sigma (\vec{E} + \vec{U}' \times \vec{B}). \tag{3}$$

The equation of continuity is

$$\vec{\nabla} \cdot \vec{U}' = 0. \tag{4}$$

The momentum equations are

$$(\vec{U}' \cdot \vec{\nabla}') \vec{U}' + 2 \vec{\Omega}' \times \vec{U}' = - \rho^{-1} \vec{\nabla}' \pi' + \nu \vec{\nabla}'^2 \vec{U}' + \rho^{-1} (\vec{J} \times \vec{B}). \tag{5}$$

Here $\pi' = p' - \frac{1}{2} \rho \mid \vec{\Omega}' \times \vec{r}' \mid^2$ and $\vec{U}', \vec{\Omega}'$ and $\vec{r}'$ are the velocity, angular velocity and position vector respectively. Also the other symbols used here have their usual meanings. μ_e is the magnetic permeability, σ the electrical conductivity and ν the kinematic viscosity of the fluid.

We choose a right handed cartesian system such that z'-axis is perpendicular to the motion of the liquid under the action of a constant pressure gradient P (i.e., $- \partial \pi' / \partial x'$) in the direction of x'-axis between two parallel porous walls $z' = \pm L$ (Stationary relative to $0' X' Y' Z'$).

Assuming that π' is independent of y' and z', π' is given by

$$- \pi' = \left(\frac{p'_1 - p'_2}{D} \right) x' - p'_1, \tag{6}$$

where p'_1 and p'_2 stand for the pressures on the planes $x' = 0$ and $x' = D$ respectively.

We suppose that the normal velocity at the wall $z' = - L$ is u'_0 ($u'_0 > 0$) so that this represents a porous wall[13] through which liquid is forced into the channel with a uniform velocity. It is further assumed that this rate of injection at the lower wall is equal to the suction rate at the upper wall. The liquid velocity is then represented by

$$\vec{U}' = \left[u'_{x'} (z'), u'_{y'} (z'), u'_0 \right], \tag{7}$$

$$\vec{\Omega}' = (0, 0, \Omega'). \tag{8}$$

Let B_0 be the intensity of the magnetic field acting perpendicular to the plates. We shall assume that the contribution to the ponderomotive force vector $\vec{J} \times \vec{B}$ of the x'-component as well as y'-component of the vector $\vec{B}$ is negligible. Furthermore we assume that the electric field in a system of reference fixed relative to the moving body is zero; this seems to be a reasonable assumption since no external electric field is applied and the effect of polarization of the conducting fluid may be expected to be small if two-dimensional conditions occur in the ionized layer[14]. With these assumptions the "retarding" magnetic forces acting on each fluid element per unit volume in the x' and y' directions are equal to $\sigma B^2_0 u'_{x'}$ and $\sigma B^2_0 u'_{y'}$ respectively.

Introducing the non-dimensional quantities

$$\vec{r}' = \vec{r} L, \; u'_x = \frac{PL^2}{2\rho\nu} u_x, \; u'_{y'} = \frac{PL^2}{2\rho\nu} u_y, \; u'_0 = \frac{\beta\nu}{L}, \; \Omega' = \frac{\nu\alpha^2}{L^2},$$

$$M = B_0 L (\sigma / \rho\nu)^{\frac{1}{2}}, \tag{9}$$

the eq. (5) reduces to

$$\beta \frac{du_x}{dz} - 2\alpha^2 u_y = 2 + \frac{d^2u_x}{dz^2} - M^2 u_x, \tag{10}$$

$$\beta \frac{du_y}{dz} + 2\alpha^2 u_x = \frac{d^2u_y}{dz^2} - M^2 u_y. \tag{11}$$

We seek the solution of eqs. (10) and (11) subject to the boundary conditions

$$u_x = u_y = 0 \text{ at } z = \pm 1. \tag{12}$$

The solution is given by

$$u_x = \frac{2 . e^{\frac{\beta z}{2}}}{(M^4 + 4\alpha^4)} \left[\frac{\cosh \frac{\beta}{2}}{\sinh^2 \frac{m}{2} + \cos^2 \frac{n}{2}} \left\{ \left(2\alpha^2 \sin \frac{n}{2} \sinh \frac{m}{2} - M^2 \cos \frac{n}{2} \cosh \frac{m}{2} \right) \cdot \cos \frac{nz}{2} \cosh \frac{mz}{2} - \left(2\alpha^2 \cosh \frac{m}{2} \cos \frac{n}{2} + M^2 \sin \frac{n}{2} \sinh \frac{m}{2} \right) \cdot \sin \frac{nz}{2} \sinh \frac{mz}{2} \right\} + \frac{\sinh \frac{\beta}{2}}{\sinh^2 \frac{m}{2} + \sin^2 \frac{n}{2}} \left\{ \left(2\alpha^2 \cos \frac{n}{2} \sinh \frac{m}{2} + M^2 \sin \frac{n}{2} \cosh \frac{m}{2} \right) \sin \frac{nz}{2} \cosh \frac{mz}{2} - \left(2\alpha^2 \sin \frac{n}{2} \cosh \frac{m}{2} - M^2 \cos \frac{n}{2} \sinh \frac{m}{2} \right) \cos \frac{nz}{2} \sinh \frac{mz}{2} \right\} \right] + \frac{2M^2}{(M^4 + 4\alpha^4)}, \tag{13}$$

$$u_y = \frac{2 . e^{\frac{\beta z}{2}}}{(M^4 + 4\alpha^4)} \left[\frac{\cosh \frac{\beta}{2}}{\sinh^2 \frac{m}{2} + \cos^2 \frac{n}{2}} \left\{ \left(2\alpha^2 \cos \frac{n}{2} \cosh \frac{m}{2} + M^2 \sin \frac{n}{2} \sinh \frac{m}{2} \right) \cdot \cos \frac{nz}{2} \cosh \frac{mz}{2} + \left(2\alpha^2 \sin \frac{n}{2} \sinh \frac{m}{2} - M^2 \cos \frac{n}{2} \cosh \frac{m}{2} \right) \cdot \sin \frac{nz}{2} \sinh \frac{mz}{2} \right\} - \frac{\sinh \frac{\beta}{2}}{\sinh^2 \frac{m}{2} + \sin^2 \frac{n}{2}} \left\{ \left(2\alpha^2 \sin \frac{n}{2} \cosh \frac{m}{2} - M^2 \cos \frac{n}{2} \sinh \frac{m}{2} \right) \sin \frac{nz}{2} \cosh \frac{mz}{2} + \left(2\alpha^2 \cos \frac{n}{2} \sinh \frac{m}{2} + M^2 \sin \frac{n}{2} \cosh \frac{m}{2} \right) \cos \frac{nz}{2} \sinh \frac{mz}{2} \right\} \right] - \frac{4\alpha^2}{(M^4 + 4\alpha^4)}, \tag{14}$$

where

$$m = \left[\frac{\sqrt{\beta^4 + 16M^4 + 8\beta^2M^2 + 64\alpha^4} + (\beta^2 + 4M^2)}{2} \right]^{\frac{1}{2}} \tag{15}$$

$$n = \left[\frac{\sqrt{\beta^4 + 16M^4 + 8\beta^2M^2 + 64\alpha^4} - (\beta^2 + 4M^2)}{2} \right]^{\frac{1}{2}}, \tag{16}$$

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For small values of β and α, we obtain

$$u_x = \Bigg[\Bigg(\frac{2}{M^2} - \frac{2\cosh Mz}{M^2\cosh M}\Bigg) + \alpha^4\Bigg(-\frac{\cosh Mz}{M^4\cosh M} + \frac{5\sinh M\cosh Mz}{M^5\cosh^2 M} + \frac{7\cosh Mz}{M^6\cosh M} + \frac{2\sinh^2 M\cosh Mz}{M^4\cosh^3 M} + \frac{z^2\cosh Mz}{M^4\cosh M} - \frac{5z\sinh Mz}{M^5\cosh M} + \frac{\cosh Mz}{M^6\cosh M} - \frac{8}{M^6} - \frac{2z\sinh M\sinh Mz}{M^4\cosh^2 M}\Bigg)\Bigg] + \beta\Bigg[\Bigg(\frac{\sinh Mz}{M^2\sinh M} - \frac{z\cosh Mz}{M^2\cosh M}\Bigg) + \alpha^4\Bigg(\frac{\sinh Mz}{6M^4\sinh M} - \frac{\cosh^2 M\sinh Mz}{M^4\sinh^3 M} + \frac{\sinh M\sinh Mz}{2M^4\cosh^2 M} + \frac{\sinh Mz}{M^5\cosh M} - \frac{5\cosh M\sinh Mz}{2M^5\sinh^2 M} - \frac{3\sinh Mz}{2M^6\sinh M} + \frac{z\cosh M\cosh Mz}{M^4\sinh^2 M} - \frac{z^2\sinh Mz}{2M^4\sinh M} + \frac{5z\cosh Mz}{2M^5\sinh M} - \frac{z^2\sinh M\sinh Mz}{2M^4\cosh^2 M} + \frac{z\sinh M\cosh Mz}{2M^5\cosh^2 M} - \frac{3z^2\sinh Mz}{2M^5\cosh M} + \frac{3z\cosh Mz}{2M^6\cosh M} + \frac{z^3\cosh Mz}{3M^4\cosh M}\Bigg)\Bigg], \tag{17}$$

$$u_y = \alpha^2\Bigg(\frac{4\cosh Mz}{M^4\cosh M} - \frac{4}{M^4} + \frac{2\sinh M\cosh Mz}{M^3\cosh^2 M} - \frac{2z\sinh Mz}{M^3\cosh M}\Bigg) + \beta\Bigg[\alpha^2\Bigg(-\frac{\cosh M\sinh Mz}{M^3\sinh^2 M} + \frac{z\cosh Mz}{M^3\sinh M} + \frac{z\sinh M\cosh Mz}{M^3\cosh^2 M} + \frac{2z\cosh Mz}{M^4\cosh M} - \frac{z^2\sinh Mz}{M^3\cosh M} - \frac{2\sinh Mz}{M^4\sinh M}\Bigg)\Bigg]. \tag{18}$$

The drag D_x in the direction of x-axis per unit area on the boundaries $z = \pm 1$ is found to be

$$\frac{2\sinh M}{M\cosh M} - \alpha^4\Bigg(\frac{3\sinh^2 M}{M^4\cosh^2 M} + \frac{3\sinh M}{M^5\cosh M} - \frac{3}{M^4} + \frac{2\sinh^3 M}{M^3\cosh^3 M} - \frac{2\sinh M}{M^3\cosh M}\Bigg) + \beta\Bigg[-\frac{\cosh M}{M\sinh M} + \frac{1}{M^2} + \frac{\sinh M}{M\cosh M} - \alpha^4\Bigg(\frac{2\cosh M}{3M^3\sinh M} - \frac{\cosh^3 M}{M^3\sinh^3 M} + \frac{2}{M^4} + \frac{3}{2M^6} - \frac{3\cosh^2 M}{2M^4\sinh^2 M} + \frac{\cosh M}{M^5\sinh M} + \frac{\sinh M}{3M^3\cosh M} - \frac{\sinh M}{M^5\cosh M} - \frac{\sinh^2 M}{2M^4\cosh^2 M}\Bigg)\Bigg]. \tag{19}$$

For large Ω^2 such that (P/α^2) remains finite and for a fixed β, we obtain from eqs. (13) and (14) for $1 \geqslant z > 0$

$$u_x \underset{-}{\Omega} \frac{1}{\alpha^2} e^{(\theta + \beta/2)(z-1)} \sin\left\{\phi(1-z)\right\}, \tag{20}$$

$$u_y \underset{-}{\Omega} \frac{1}{\alpha^2}\left[e^{(\theta+\beta/2)(z-1)} \cos\left\{\phi(1-z)\right\} - 1\right] \tag{21}$$

and for $0 > z \geqslant -1$,

$$u_x \underset{-}{\Omega} \frac{1}{\alpha^2} e^{-(\theta - \beta/2)(1+z)} \sin\left\{\phi(1+z)\right\}, \tag{22}$$

$$u_y \underset{-}{\Omega} \frac{1}{\alpha^2}\left[-1 + e^{-(\theta-\beta/2)(1+z)} \cos\left\{\phi(1+z)\right\}\right], \tag{23}$$

where

$$\theta^2 - \phi^2 = M^2, \quad \theta\phi = \alpha^2$$

Putting $\beta=0$ in eqs. (20, 21), we recover eqs. (3·11) of Vidyanidhi[2]. Similarly when $M = 0$ in eqs. (20 to 23), we recover eqs. (2·17 to 2·20) of our earlier work[12].

NUMERICAL CALCULATIONS AND DISCUSSION

Symmetry of the Flow

If we replace the suction Reynolds number β by their negative values and z by $-z$, the expressions for both the primary and secondary velocity distributions as given by eqs. (13) and (14) respectively do not change. This shows that when there is uniform injection at the lower wall, the primary and the secondary flow distributions in the lower half are the same as in the upper half for the case of uniform injection at the upper wall and vice-versa. Hence in the numerical computations involved in this problem, we have considered the positive values of the suction Reynolds numbers.

The normalized velocity distributions for the primary, secondary flows, when $\alpha=0{\cdot}841$, $\beta=1$ and $M=0, 2$ have been shown in Figs. 1 and 2 to illustrate the effect of porosity. For constant suction Reynolds numbers, the velocity profiles tend to the symmetrical position about the central line with increasing values of Hartmann number M, as in the case of Hartmann[15] where the suction and injection are absent. But for constant Hartmann number, as we increase the value of β, this sort of behaviour is slender.

Boundary Layers for Large α and Finite M and β

We note from eq. (20) that the amplitude of u_∞ is positive and that the function $\sin\{\phi\,(1 - z)\}$ can take positive or negative values. For $\alpha \to \infty$, such that (P/α^2) is finite, the disturbance is confined to regions of order $L/(\theta+\beta/2)$ in the vicinity of the suction wall and $L/(\theta-\beta/2)$ in the vicinity of the injection wall, the thickness of the boundary layers being of order

$$\left(\frac{u_0'}{2\nu} + \sqrt{\frac{\Omega'}{\nu} + \frac{\mu_e^2 H_0^2}{2\rho\nu}}\right)^{-1} \text{ and } \left(-\frac{u'_0}{2\nu} + \sqrt{\frac{\Omega'}{\nu} + \frac{\mu_e^2 H_0^2}{2\rho\nu}}\right)^{-1}$$

This shows that in the absence of the magnetic field, suction causes thinning of the boundary layer[2], while injection causes a thickening of the boundary layer[9,16]. The effect of the magnetic field on a rapidly rotating system is to reduce the thickness of the boundary layer at both the suction and injection walls and more at the injection wall than that the suction wall. This is pronounced from the nature of graphs of Figs. 1 and 2 close to the vicinity of walls.

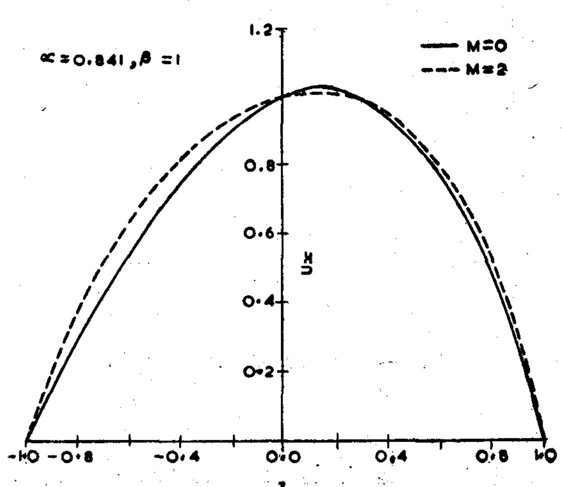

Fig. 1—Normalized velocity rofiles of the primary flow.

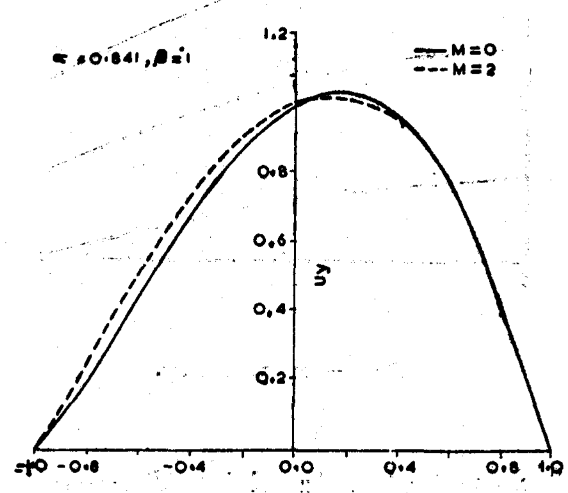

Fig. 2—Normalized velocity profiles of the secondary flow.

Stream Lines

The stream lines can be examined at this stage. In the non-magnetic case the stream lines are no longer confined to planes parallel to the walls implying that a particle of liquid once in a plane leaves it in its subsequent motion. Due to uniform suction and injection, the effect of the magnetic field is to retard and confine the stream lines to planes parallel to the walls implying that a particle of liquid once in a plane does not leave it in its subsequent motion. For small values of α, β and M the angle ϵ, which the projection of the stream lines makes with the x-axis decreases from

$$\tan^{-1}\left[-\alpha^2\left(\frac{5}{6}-\frac{119M^2}{360}\right)\right] \text{ at } z=0 \text{ to}$$

$$\tan^{-1}\left[-\alpha^2\left\{\frac{2}{3}-\frac{14M^2}{45}+\beta\left(\frac{4}{45}-\frac{8M^2}{315}\right)\right\}\right] \text{ at } z=1 \text{ and to}$$

$$\tan^{-1}\left[-\alpha^2\left\{\frac{2}{3}-\frac{14M^2}{45}-\beta\left(\frac{4}{45}-\frac{8M^2}{315}\right)\right\}\right] \text{ at } z=-1$$

It is found for fixed α and β, ϵ decreases as M increases.

Drag, Mass Flow Rate and the Resistance Coefficient

Fig. 3 shows the calculated values of the drag at either wall and based on eq. (19). It is concluded that for fixed values of α and β, the effect of the magnetic field is to decrease the drag at both the walls while for fixed α and M, the drag at both the walls increases as β increases[16].

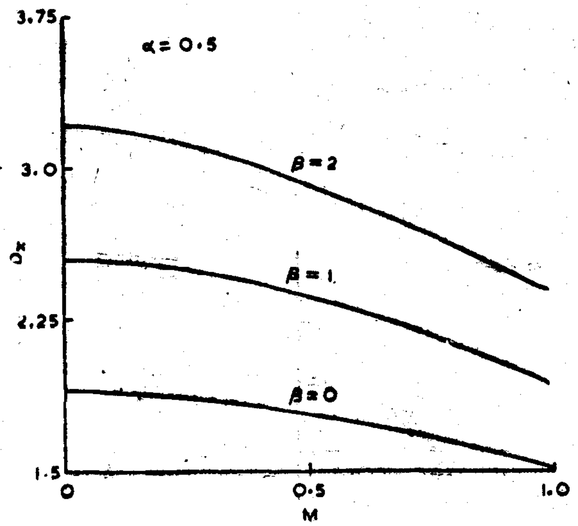

Fig. 3—Drag at the walls.

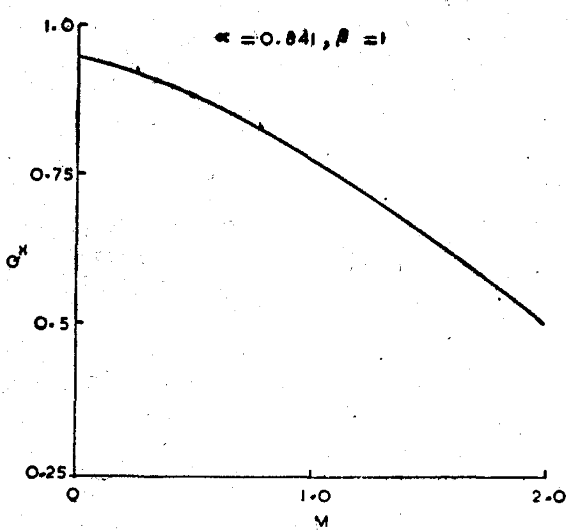

Fig. 4—Mass flow rate along x-axis.

For fixed values of α and β, as M increases, we observed (i) from Fig. 4, the mass flow rate along the x-axis decreases (ii) from the table given below the resistance coefficient[17] (Γ_r/Γ) or $(Q/Q_x)^2$ where Q and Q_x denote the flux for flows without and with rotation, decreases,

TABLE 1

VALUES OF THE RESISTANCE COEFFICIENT FOR $\alpha = 0{\cdot}841,\ \beta = 1$

M	0	1	2
Γ_r/Γ	1·613	1·397	1·111

Suggestions for Experimental Verification

It may be possible to perform experiments by rotating a channel of finite width B which is great compared with the depth $2L$. In such a channel the conditions close to the walls $z'=\pm L$ are not given by the above calculations but if the side walls are in such a direction that there is no total flow across them, then the conditions can be attained approximately over most of the channel. It is necessary to keep the side walls at an angle '—A' with x'-axis where

$$\tan A = -\int_{-1}^{1} u_y \, d_z \Big/ \int_{-1}^{1} u_x \, d_z$$

It is concluded from Fig. 5 that for fixed α and β, this angle A decreases as M increases. The effect of the magnetic field is to inhibit the secondary flow through the side walls.

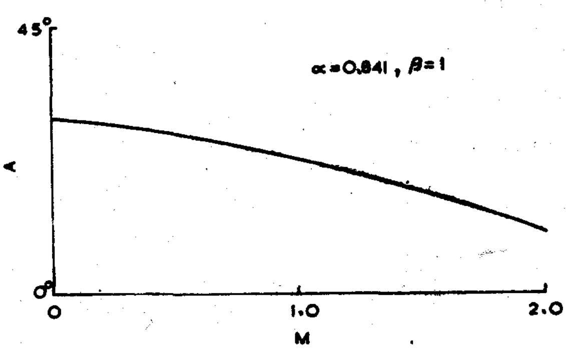

Fig. 5—The angle A as a function of the parameter M.

REFERENCES

1. VIDYANIDHI, V. & NIGAM, S.D., *Jour. Math. Phy. Sci.*, **1** (1967), 85.

2. VIDYANIDHI, V., *Jour. Math. Phy. Sci.*, **3** (1969), 193.

3. BERMAN, A.S., *J. Appl. Phys.*, **24** (1953), 1232.

4. SURYA PRAKASA RAO, U., *Appl. Sci. Res.*, **B9** (1962), 374.

5. TERRILL, R.M. & SHRESTHA, G.M., *Appl. Sci. Res.*, **B11** (1964), 134.

6. REDDY, P.G. & JAIN, M.K., *J. Phys. Soc. Japan*, **22** (1967), 1255.

7. VIDYANIDHI, V. & RAMANA RAO, V.V., *Proc. Ind. Acad. Sci. A.*, **72** (1970), 272

8. STUART, J. T., *Quart. J. Mech.*, **7** (1954), 446.

9. GUPTA, A.S., *Phys. Fluids*, **15** (1972), 930,

10. Debnath, L. & Mukherjee, S., *Phys. Fluids*, 16 (1973), 1418.

11. Singh, M.P. & Sathi, H.L., *Jour. Math. and Mechs.*, 18 (1968), 193.

12. Vidyanidhi, V., Bala Prasad, V. & Ramana Rao, V.V., *J. Phys. Soc. Japan*, 39 (1975), 1077.

13. Batchelor, G.K., "An Introduction to Fluid Dynamics". (Camb. Univ. Press), 1967, p. 284.

14. E.L. Resler, J.R. & Sears, W.R., *J. Aero. Sci.*, 25 (1958), 235.

15. Hartmann, J., *Math. Phys. Medd.*, 15 (1937), 6.

16. Rosenhead, L., "Laminar Boundary Layers". (Oxford Univ. Press), 1963, pp. 333, 334.

17. Goldstein, S., "Modern Developments in Fluid Dynamics". Edited (Oxford Univ.), 1938, Ist Edition.