A PROBLEM IN THE GENERALISED THEORY OF THERMO-ELASTICITY

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(*Received 6 October 1976; revised, 16 March 1977*)

This paper deals with the dynamic treatment of a transient thermo-elastic half space that is exposed to step temperature and velocity on its entire plane boundary and is constrained against transverse displacements, using generalized theory of thermo-elasticity.

Aircraft structural designers usually deal with the thermal stress problems associated with elevated temperatures in airplane, missile structures, jet engines and nuclear reactors. The thermal-stress distribution pertaining to the problem whose plane boundry is subjected to step and/or ramp type temperatures only, was treated by V. I. Danilovskaya[1], T. Mura[2] and E. Sternburg & J. G. Chakrabarthy[3]. Physically, when the boundary experiences temperature variation, it often also experiences velocity variations. Y. T. Tsui[4], showed that the thermal-stress distribution changes considerably once the velocity variation on the boundary is considered. He also determined thermal-stress distribution and displacement distribution when the half space is exposed to step temperature and velocity, using uncoupled theory.

The aim of this paper is to determine temperature distribution and the subsequent thermal-stress distribution when the half space is exposed to step-temperature and velocity, using the generalized theory of thermo-elasticity which takes into account the effect of relaxation time.

GOVERNING EQUATIONS

The energy equation is

$$k\, T_{,ii} = \rho C_E\,(\dot{T} + \tau_0 \ddot{T}) + (3\lambda + 2\mu)\,\alpha T_0\,(\dot{\epsilon}_{KK} + \tau_0\, \ddot{\epsilon}_{KK}) \tag{1}$$

The equation of motion is

$$\rho \ddot{u}_i = (\lambda + \mu)\, u_{j,ij} + \mu u_{i,jj} - (3\lambda + 2\mu)\,\alpha T_{,i} \tag{2}$$

The stress-strain, temperature relation is

$$\sigma_{ij} = \lambda\, \epsilon_{KK}\, \delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\,\alpha\,(T - T_0)\,\delta_{ij} \tag{3}$$

Now using the following non-dimensional variables

$$Z = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} \frac{\rho C_E}{k}\, x\,,\ \tau = \left(\frac{\lambda + 2\mu}{\rho}\right) \frac{\rho C_E t}{k}$$

$$\theta = \frac{T - T_0}{T_0}\,,\ \Sigma = \frac{\sigma}{(3\lambda + 2\mu)\alpha T_0} \tag{4}$$

$$U = \left[\rho\left(\frac{\lambda + 2\mu}{\rho}\right)^{3/2} \frac{1}{(3\lambda + 2\mu)\,\alpha T_0}\, \frac{\rho C_E}{k}\right] u$$

Equations (1), (2) and (3) become

$$\theta'' - \dot{\theta} - \beta\,\ddot{\theta} = e\,(\dot{U}' + \beta \ddot{U}') \tag{5}$$

$$U'' - \ddot{U} = \theta' \tag{6}$$

$$\Sigma = U' - \theta \tag{7}$$

'′' denotes differentiation with respect to Z and

'·' denotes differentiation with respect to τ.

where

λ, μ are Lami's constants,

k = Thermal conductivity,

α = Coefficient of Linear expansion,

τ_0 = Relaxation time,

C_E = Specific heat at constant deformation,

$$\left.\begin{aligned} e &= \text{Thermo-elastic coupling constant} = \frac{(3\lambda + 2\mu)^2 \alpha^2 T_0}{(\lambda + 2\mu)\,\rho\, C_E} \\ &\text{and} \\ \beta &= \text{Relaxation Constant} = \frac{(\lambda + 2\mu)}{\rho}\left(\frac{\rho\, C_E}{k}\right)\tau_0 \end{aligned}\right\} \quad (8)$$

FORMULATION AND SOLUTION OF THE PROBLEM

Consider an elastic half-space whose boundary $x=0$ is subject to a step temperature and step velocity. The problem is to find the temperature distribution in the half-space $x > 0$ with constrained lateral displacements i.e. $v = w = 0$.

i.e. to solve the equations

$$\theta'' - \dot{\theta} - \beta\ddot{\theta} = e(\dot{U}' + \beta\ddot{U}')$$

$$U'' - \ddot{U} = \theta'$$

$$\Sigma = U' - \theta$$

subject to the following boundary conditions

$$\left.\begin{aligned} \theta(z, \tau) &= H(\tau), \text{ when } Z = 0. \\ \left.\frac{\partial U}{\partial \tau}\right|_{Z=0} &= H(\tau) \end{aligned}\right\} \quad (9)$$

where $H(\tau)$ is the heaviside step function.

$$H(\tau) = 0 \text{ for } \tau < 0$$

$$= 1 \text{ for } \tau > 0$$

We can neglect the strain acceleration term i e. $\ddot{U}'$ in the energy equation, since the product $e\beta$ is much less than either e or β in the intermediate range and room temperature[5].

The problem to be solved is

$$\left.\begin{aligned} \theta'' - \dot{\theta} - \beta\ddot{\theta} &= e\dot{U}' \\ U'' - \ddot{U} &= \theta' \\ \Sigma &= U' - \theta \end{aligned}\right\} \quad (10)$$

subject to the boundary conditions (9).

The initial conditions are taken as

$$\theta(Z, 0) = 0, \qquad U(Z, 0) = U'(Z, 0) = 0, \quad (11)$$

From (10), we can get after eliminating U

$$\theta'''' - (1+\beta)\ddot{\theta}'' - (1+e)\dot{\theta}'' + \dddot{\theta} + \beta\ddddot{\theta} = 0 \tag{12}$$

Similary by eliminating θ, we get

$$U'''' - (1+\beta)\ddot{U}'' - (1+e)\dot{U}'' + \beta\ddddot{U} + \dddot{U} = 0 \tag{13}$$

Now applying Laplace transform with respect to τ, we get

$$\bar{U}'''' - p[p(1+\beta) + (1+e)]\bar{U}'' + p^2(p+\beta p^2)\bar{U} = 0 \tag{14}$$

where

$$\bar{U}(Z,p) = \int_0^\infty \operatorname{Exp}(-p\tau)\, U(Z,\tau)\, d\tau \tag{15}$$

and p is the transform parameter.

The transformed boundary conditions will take the form

$$\bar{\theta}(Z,p) = \frac{1}{p}, \qquad \overline{\frac{\partial U(Z,\tau)}{\partial \tau}} = p\bar{U} = \frac{1}{p} \tag{16}$$

Similarly

$$\bar{\theta}'''' - p[p(1+\beta) + (1+e)]\bar{\theta}'' + p^2(p+\beta p^2)\theta = 0 \tag{17}$$

Now using the regularity boundary conditions, i.e.

$$\underset{Z\to\infty}{\mathrm{Lt}}\ \bar{\theta}(Z,p) = \underset{Z\to\infty}{\mathrm{Lt}}\ \bar{U}(Z,p) \longrightarrow \text{finite value.}$$

We get

$$\bar{\theta}(Z,p) = A\operatorname{Exp}(-\alpha_1 Z) + B\operatorname{Exp}(-\alpha_2 Z) \tag{18}$$

and

$$\bar{U}(Z,p) = C\operatorname{Exp}(-\alpha_1 Z) + D\operatorname{Exp}(-\alpha_2 Z) \tag{19}$$

where

$$\alpha_{1,2} = (p/2)^{1/2}\left[p(1+\beta) + 1 + e \pm \left[[p(1+\beta) + (1+e)]^2 - 4(p+\beta p^2)\right]^{1/2}\right]^{1/2} \tag{20}$$

A,B,C,D are constants.

Now taking the series expansion for large values of p (i.e. for small values of time) and neglecting the negative powers[6] of p, we get

$$\alpha_1 = p + e/2 \tag{21}$$

$$\alpha_2 = \sqrt{\beta}\, p + 1/2\sqrt{\beta} \tag{22}$$

Using the boundary conditions (16) and with the two equations obtained by substituting (18) and (19) in the first and second equations of (10) the constants A, B, C, D, are found to be

$$A = -\frac{e\sqrt{\beta}\, p^2 + \left[\frac{1}{4\beta} - \frac{1}{4} - \frac{e}{2\sqrt{\beta}}\right]p + \left[\frac{1}{4\beta}(e-1) + e(1-2\sqrt{\beta})\right]}{p(p-A_1)(p-A_2)(p-A_3)} \tag{23}$$

$$B = \frac{1}{p} - A \tag{24}$$

$$C = \frac{\left[1+\beta\,(\sqrt{\beta}-2)\right] p^2 + \left[2(\beta-1)+e+3\sqrt{\beta}/2-1/2\sqrt{\beta}\right] p + \left[\frac{3}{2}-2e+\frac{1}{2\sqrt{\beta}}-\frac{1}{4\beta}\right]}{p\,(p-A_1)\,(p-A_2)\,(p-A_3)} \tag{25}$$

and

$$D = \frac{1}{p^2} - C \tag{26}$$

where

$$(1-2\beta)\,p^3 + [\,2\,(\beta-1+e\sqrt{\beta})+e\,]\,p^2 + [3/2-3e-1/2\beta+e/\sqrt{\beta}\,]\,p + $$
$$+ [\,(1/2\beta)\,(1-3e/2)\,] = (p-A_1)\,(p-A_2)\,(p-A_3) \tag{27}$$

$$\therefore\ \bar{\theta}\,(Z,p) = \text{Exp}\,(-Z/2\sqrt{\beta})\,\frac{\text{Exp}\,(-\sqrt{\beta}\,Zp)}{p} + \text{Exp}\,(-Z/2\sqrt{\beta})\,\frac{\text{Exp}\,(-\sqrt{\beta}\,Zp)}{p} \times$$
$$\times\left[\frac{e\sqrt{\beta}\,p^2 + [\,1/4\beta-1/4-e/2\sqrt{\beta}\,]\,p + [\,(1/4\beta)\,(e-1)+e\,(1-2\sqrt{\beta})\,]}{(p-A_1)\,(p-A_2)\,(p-A_3)}\right] -$$
$$-\,\text{Exp}\,(-eZ/2)\,\frac{\text{Exp}\,(-pZ)}{p}\left[\frac{e\sqrt{\beta}p^2 + \left[\frac{1}{4\beta}-\frac{1}{4}-\frac{e}{2\sqrt{\beta}}\right]p + \left[\frac{1}{4\beta}\,(e-1)+e\,(1-2\sqrt{\beta})\right]}{(p-A_1)\,(p-A_2)\,(p-A_3)}\right] \tag{28}$$

Taking the inverse transform[7], we get

$$\theta(Z,\tau) = 0, \qquad \text{when } 0<\tau<\sqrt{\ }Z$$

$$= \text{Exp}\,(-Z/2\sqrt{\beta})\,\Big[1+F(A_1)\,[\text{Exp}\,(A_1\tau)-1] + F(A_2)\,[\text{Exp}\,(A_2\tau)-1] +$$
$$+ F(A_3)[\,\text{Exp}\,(A_3\tau)-1\,]\Big] \qquad \text{when } \sqrt{\beta}Z<\tau<Z. \tag{29}$$

$$= \Big[\,\text{Exp}\,(-Z/2\sqrt{\beta}) - \text{Exp}\,(-eZ/2)\,\Big]\Big[\,F(A_1)\,[\,\text{Exp}\,(A_1\tau)-1\,] +$$
$$+ F(A_2)\,[\,\text{Exp}\,(A_2\tau)-1\,] + F(A_3)\,[\,\text{Exp}\,(A_3\tau)-1\,]\Big]$$
$$+\,\text{Exp}\,(-Z/2\sqrt{\beta}) \qquad \text{when } \tau > Z$$

where

$$\left.\begin{aligned} F(A_1) &= M(A_1)\,/\,A_1\,(A_1-A_2)\,(A_1-A_3) \\ F(A_2) &= M(A_2)\,/\,A_2\,(A_2-A_1)\,(A_2-A_3) \\ F(A_3) &= M(A_3)\,/\,A_3\,(A_3-A_1)\,(A_3-A_2) \end{aligned}\right\} \tag{30}$$

and

$$M(A_i) = e\sqrt{\beta}\,A_i^2 + [\,1/4\beta-1/4-e/2\sqrt{\beta}\,]\,A_i + [(1/4\beta)\,(e-1)+e\,(1-2\sqrt{\beta})]$$
$$\text{for } i = 1, 2, 3. \tag{31}$$

$$\bar{U}\,(Z,p) = \text{Exp}\,(-Z/2\sqrt{\beta})\,\frac{\text{Exp}\,(-\sqrt{\beta}\,pZ)}{p_2} +$$
$$+\left[\frac{[1+\beta(\sqrt{\beta}-2)]\,p^2 + [2(\beta-1)+e+3\sqrt{\beta}/2-1/2\sqrt{\beta}\,]\,p + [3/2-2e+1/2\sqrt{\beta}-1/4\beta\,]}{(p-A_1)\,(p-A_2)\,(p-A_3)}\right] \times$$
$$\times\left[\,\text{Exp}\,(-e\,Z/2\,\frac{\text{Exp}\,(-pZ)}{p} - \text{Exp}\,(-Z/2\sqrt{\beta})\,\frac{\text{Exp}\,(-\sqrt{\beta}\,pZ)}{p}\right] \tag{32}$$

Taking the inverse transform[7], we get

$$U(Z,\tau) = 0 \qquad \text{when } 0 < \tau < \sqrt{\beta}\, Z$$

$$= \text{Exp}\,(-Z/2\sqrt{\beta})\Big[\tau - \Big[G(A_1)\,[\text{Exp}\,(A_1\tau) - 1] + G(A_2)\,[\text{Exp}\,(A_2\tau) - 1] + G(A_3)\,[\text{Exp}\,(A_3\tau) - 1]\Big]\Big] \qquad \text{when } \sqrt{\beta}\, Z < \tau < Z$$

$$= \Big[\text{Exp}\,(-eZ/2) - \text{Exp}\,(-Z/2\sqrt{\beta})\Big]\Big[G(A_1)\,[\text{Exp}\,(A_1\tau) - 1] + G(A_2)\Big[\text{Exp}\,(A_2\tau) - 1\Big] + G(A_3)\,[\text{Exp}\,(A_3\tau) - 1]\Big] + \tau\,\text{Exp}\,(-Z/2\sqrt{\beta}) \qquad \text{when } \tau > Z \tag{33}$$

where

$$\left.\begin{aligned} G(A_1) &= N(A_1)\,/\,A_1(A_1 - A_2)\,(A_1 - A_3) \\ G(A_2) &= N(A_2)\,/\,A_2(A_2 - A_1)\,(A_2 - A_3) \\ G(A_3) &= N(A_3)\,/\,A_3(A_3 - A_1)\,(A_3 - A_2) \end{aligned}\right\} \tag{34}$$

and

$$N(A_i) = [1 + \beta(\sqrt{\beta} - 2)]\,A_i^2 + [2(\beta - 1) + e + 3\sqrt{\beta}/2 - 1/2\sqrt{\beta}]\,A_i + [3/2 - 2e + 1/2\sqrt{\beta} - 1/4\beta] \quad \text{for } i = 1, 2, 3 \tag{35}$$

The stress distribution is

$$\Sigma = U' - \theta$$

$$\therefore\ \Sigma = 0 \qquad \text{when } 0 < \tau < \sqrt{\beta}\, Z$$

$$= -(1/2\sqrt{\beta})\,\text{Exp}\,(-Z/2\sqrt{\beta})\Big[\tau - \Big[G(A_1)[\text{Exp}\,(A_1\tau) - 1] + G(A_2)[\text{Exp}\,(A_2\tau) - 1] + G(A_3)[\text{Exp}\,(A_3\tau) - 1]\Big]\Big] - \text{Exp}\,(-Z/2\sqrt{\beta})\Big[F(A_1)[\text{Exp}\,(A_1\tau) - 1] + F(A_2)[\text{Exp}\,(A_2\tau) - 1] + F(A_3)[\text{Exp}\,(A_3\tau) - 1]\Big] \qquad \text{when } \sqrt{\beta}\, Z < \tau < Z$$

$$= -(1 + \tau/2\sqrt{\beta})\,\text{Exp}\,(-Z/2\sqrt{\beta}) + \Big[(1/2\sqrt{\beta})\,\text{Exp}\,(-Z/2\sqrt{\beta}) - (e/2)\,\text{Exp}\,(-e\,Z/2)\Big]\Big[G(A_1)[\text{Exp}\,(A_1\tau) - 1] + G(A_2)[\text{Exp}\,(A_2\tau) - 1] + G(A_3)[\text{Exp}\,(A_3\tau) - 1]\Big] + \Big[\text{Exp}\,(eZ/2) - \text{Exp}\,(-Z/2\sqrt{\beta})\Big]\Big[F(A_1)[\text{Exp}\,(A_1\tau) - 1] + F(A_2)[\text{Exp}\,(A_2\tau) - 1] + F(A_3)[\text{Exp}\,(A_3\tau) - 1]\Big] \qquad \text{when } \tau > Z. \tag{36}$$

Which is the required stress distribution. This completes the solution of the problem.

ACKNOWLEDGEMENT

The author wishes to thank Dr. M. G. K. Murthy, for his valuable suggestions during this work.

REFERENCES

1. Danilovskaya, V. I., *Prikl. Mat. I. Mechan*, (1950), p. 316.
2. Mura, T., *Proc. Japan. Natt. Cong. App. Mech.*, (1952).
3. Sternburg, E. & Chakrabarthy, J. G., *J. App. Mech.*, 26 (1959).
4. Tsui, Y. T., *J. Accoust. Soc. Ame.*, (1966), 176.
5. Lord, H. W. & Shulman, Y., *J. Mech. Phys. Solids*, 15 (1967), 299.
6. Massashi Daimaruya, Masachika Naito, *Bull. of the J.S.M.E.*, 16 (1973), 1494.
7. Bateman Manuscript Project "Tables of integral Transforms" Vol. 1 (McGraw Hill, New York), 1954.