BENDING OF AN ISOTROPIC COMPRESSIBLE CIRCULAR PLATE INTO AN ELLIPSOIDAL SHELL

RAM KUMAR SHUKLA

Military College of Electronics and Mechanical Engineering, Secunderabad

&

G. LAKSHMINARAYANA

Nagarjunasagar Engineering College, Hyderabad

(*Received* 22 *September* 1976)

The problem of bending of an isotropic compressible circular plate into an ellipsoidal shell has been solved and the results obtained in terms of the general strain energy function w.

One of the major advances of the century in the theory of finite deformations has been made by Rivlin[1], Green, Zerna and Adkins[2 & 3] by obtaining a number of exact solutions, specially for incompressible bodies in terms of general strain energy function.

Recently problems of finite bending of compressible circular plates into spherical and ellipsoidal shells have been discussed by Seth[4 & 5] and Lakshminarayana[6 & 7] on the basis of quasi linear stress strain relations proposed by Seth[8].

In this paper an attempt has been made to solve the problem of bending of an isotropic compressible circular plate into an ellipsoidal shell and obtain the solutions in terms of general strain energy function. The cases of compressible place bent into spherical shell and incompressible plates bent into ellipsoidal and spherical shells have been derived as particular cases of the problem solved.

NOTATIONS AND FORMULAE

Following the notations of Green & Zerna[2], we take X_i-axes to define the unstrained body B_0 and Y_i-axes to refer the strained body B. We take the curvilinear coordinates θ^i in the strained body. Let the covariant and contravariant metric tensors in the deformed and the undeformed body be respectively G_{ij}, G^{ij} and g_{ij}, g^{ij}. When the unstrained body is homogeneous and isotropic, the strain energy function w measured per unit volume of the unstrained body is a function of strain invariants so that

$$w = w \ (I_1, I_2, I_3) \tag{1}$$

where

$$I_1 = g^{rs} G_{rs},\ I_2 = g_{rs} G^{rs} I_3,\ I_3 = \frac{G}{g} \tag{2}$$

and

$$G_{rs} = \frac{\partial y^\alpha}{\partial \theta^r} \frac{\partial y^\alpha}{\partial \theta^s},\ G = | G_{ij} | \tag{3}$$

$$g_{rs} = \frac{\partial x^\alpha}{\partial \theta^r} \frac{\partial x^\alpha}{\partial \theta^s},\ g = | g_{ij} | \tag{4}$$

The contravariant stress tensor measured per unit area of strained body referred to θ^i coordinates is given by

$$\tau^{ij} = g^{ij} \Phi + B^{ij} \Psi + G^{ij} p \tag{5}$$

where

$$\Phi = \frac{2}{\sqrt{I_3}} \frac{\partial w}{\partial I_1},\ \psi = \frac{2}{\sqrt{I_3}} \frac{\partial w}{\partial I_2},\ p = 2\sqrt{I_3} \frac{\partial w}{\partial I_3} \tag{6}$$

$$B^{ij} = \frac{1}{g} e^{irm} e^{jsn} g_{rs} G_{mn} \tag{7}$$

The equations of equilibrium are given by

$$\tau^{ik}{}_{,i} + \Gamma^i_{ir} \tau^{rk} + \Gamma^k_{ir} \tau^{ir} + \rho F^K = 0 \tag{8}$$

where

$$\Gamma^{i}_{jk} = \frac{1}{2} G^{is} (G_{sj,k} + G_{sk,j} - G_{jk,s}) \tag{9}$$

comma denotes partial differentiation with respect to the strained body and F^k are the components of the body force and ρ the density.

The physical components of the stress σ_{ij} are given by

$$\sigma_{ij} = \sqrt{\frac{G_{jj}}{G^{ii}}}\ \tau^{ij} \tag{10}$$

BENDING OF THE BLOCK

Suppose that in the undeformed state of the body, it is a circular block bounded by the planes $x_3 = a_2$ and $x_3 = a_1$, $a_2 > a_1$, and the cylinder. $x_1^2 + x_2^2 = a^2$.

The block is then bent symmetrically about x_3 + axis into a part of an ellipsoidal shell, whose inner and outer boundaries are the ellipsoides of revolution obtained by revolving the confocal ellipses

$$x_3 = c \cosh \xi \cos \eta,\ y_3 = c \sinh \xi \sin \eta,\ \xi = \xi_i,\ i = 1, 2 \tag{11}$$

about the x_3-axis respectively and the edge $\eta = \alpha$. Let the y_i-axis coincide with the x_i-axis and the curvilinear coordinates θ^i in the deformed state be a system of orthogonal curvilinear coordinates (ξ, η, ϕ) where ϕ is the angle between $y_1\, y_2$ plane and the plane through a point in space and y_3-axis, then

$$y_1 = c \sinh \xi \sin \eta \cos \phi,\ y_2 = c \sinh \xi \sin \eta \sin \varphi,\ y_3 = c \cosh \xi \cos \eta \tag{12}$$

since the deformation is symmetrical about y_3-axis, we see that :

(i) the planes x_3 = constant in the undeformed state become the ellipsoidal surfaces ξ = const in the deformed state,

(ii) the curves $x_1^2 + x_2^2$ = constant, in the undeformed state become the circles η = constant, in the deformed state,

(iii) are $\tan \dfrac{x_2}{x_1} = \phi$

These imply[7]

$$x_3 = f(\xi),\quad x_1^2 + x_2^2 = F(\eta) = \lambda\eta \tag{13}$$

which give

$$x_1 = \lambda\eta \cos \phi,\quad x_2 = \lambda\eta \sin \phi,\quad x_3 = f(\xi) \tag{14}$$

The metric tensors for the strained and the unstrained states of the body are given by

$$G_{ij} = \begin{bmatrix} c^2 (\cosh^2 \xi - \cos^2 \eta) & 0 & 0 \\ 0 & c^2 (\cosh^2 \xi - \cos^2 \eta) & 0 \\ 0 & 0 & c^2 \sinh^2 \xi \sin^2 \eta \end{bmatrix} \tag{15}$$

$$G^{ij} = \begin{bmatrix} \dfrac{1}{c^2 (\cosh^2 \xi - \cos^2 \eta)} & 0 & 0 \\ 0 & \dfrac{1}{c^2 (\cosh^2 \xi - \cos^2 \eta)} & 0 \\ 0 & 0 & \dfrac{1}{c^2 (\sinh^2 \xi \sin^2 \eta)} \end{bmatrix} \tag{16}$$

$$g_{ij} = \begin{bmatrix} f_\xi^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2\eta^2 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \frac{1}{f_\xi^2} & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^2\eta^2} \end{bmatrix}, \quad g = \lambda^4 \eta^2 f_\xi^2 \tag{17}$$

Where $f_\xi = \dfrac{df}{d\xi}$. Here we consider the case when η is so small[7] that $\sin \eta$ is replaced by η and $\cos \eta$ is replaced by unity. Then (15) and (16) become

$$G_{ij} = \begin{bmatrix} c^2 \sinh^2 \xi & 0 & 0 \\ 0 & c^2 \sinh^2 \xi & 0 \\ 0 & 0 & c^2 \eta^2 \sinh^2 \xi \end{bmatrix}, \quad G^{ij} = \begin{bmatrix} \frac{1}{c^2 \sinh^2 \xi} & 0 & 0 \\ 0 & \frac{1}{c^2 \sinh^2 \xi} & 0 \\ 0 & 0 & \frac{1}{c^2 \eta^2 \sinh^2 \xi} \end{bmatrix} \tag{18}$$

$$G = c^6 \eta^2 \sinh^6 \xi \tag{19}$$

Substituting (17), (18) in (2), we have

$$I_1 = \frac{c^2 \sinh^2 \xi}{f_\xi^2} + \frac{2c^2 \sinh^2 \xi}{\lambda^2}, \quad I_2 = \frac{c^4 \sinh^4 \xi}{\lambda^4 f_\xi^2}\left(f_\xi^2 + 2\lambda^2\right), \quad I_3 = \frac{c^6 \sinh^6 \xi}{\lambda^4 f_\xi^2} \tag{20}$$

Substituting (17) and (18) in (7), we have

$$B^{ij} = \frac{c^2 \sinh^2 \xi}{\lambda^2} \begin{bmatrix} \frac{2}{f_\xi^2} & 0 & 0 \\ 0 & \frac{1}{\lambda^2} + \frac{1}{f_\xi^2} & 0 \\ 0 & 0 & \frac{1}{\eta^2}\left(\frac{1}{\lambda^2} + \frac{1}{f_\xi^2}\right) \end{bmatrix} \tag{21}$$

The non-vanishing components of the stress (5) are given by

$$\left.\begin{aligned} \tau^{11} &= \frac{\phi}{f_\xi^2} + \frac{2c^2 \sinh^2 \xi}{\lambda^2 f_\xi^2}\psi + p \\ \tau^{22} &= \frac{\phi}{\lambda^2} + \frac{c^2 \sinh^2 \xi}{\lambda^2}\left(\frac{1}{\lambda^2} + \frac{1}{f_\xi^2}\right)\psi + \frac{p}{c^2 \sinh^2 \xi} \\ \tau^{33} &= \frac{\phi}{\lambda^2\eta^2} + \frac{c^2 \sinh^2 \xi}{\lambda^2\eta^2}\left(\frac{1}{\lambda^2} + \frac{1}{f_\xi^2}\right)\psi + \frac{p}{c^2 \eta^2 \sinh^2 \xi} \end{aligned}\right\} \tag{22}$$

The physical components of the stress (10) are given by

$$\sigma_{11} = c^2 \sinh^2 \xi \, \tau^{11}, \quad \sigma_{22} = c^2 \sinh^2 \xi \, \tau^{22}, \quad \sigma_{33} = c^2 \eta^2 \sinh^2 \xi \, \tau^{33} \quad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \tag{23}$$

Substituting (18) in (9), the non-zero christofel's symbols are given by

$$\Gamma_{11}^{\ 1} = \Gamma_{21}^{\ 2} = \Gamma_{31}^{\ 3} = \coth \xi, \ \Gamma_{22}^{\ 1} = -\coth \xi \tag{24}$$

$$\Gamma_{33}^{\ 1} = -\eta^2 \coth \xi, \ \Gamma_{33}^{\ 2} = -\eta, \ \Gamma_{32}^{\ 3} = \frac{1}{\eta} \tag{25}$$

In the absence of the body forces, the equations of equilibrium to be satisfied are

$$\frac{\partial \sigma_{11}}{\partial \xi} + 2 \coth \xi \, (\sigma_{11} - \sigma_{22}) = 0 \tag{26}$$

$$\frac{\partial p}{\partial \eta} = 0 \tag{27}$$

$$\frac{\partial p}{\partial \phi} = 0 \tag{28}$$

From (27) and (28) it is obvious that p is purely a function of ξ only. Since the strain energy function w is a function of I_1, I_2, I_3, and in turn I_1, I_2, I_3 are functions of ξ only, we have

$$\frac{dw}{d\xi} = \frac{\partial w}{\partial I_1}\frac{dI_1}{d\xi} + \frac{\partial w}{\partial I_2}\frac{dI_2}{d\xi} + \frac{\partial w}{\partial I_3}\frac{dI_3}{d\xi} \tag{29}$$

From (20) and (6) we have

$$\frac{\partial w}{\partial I_1} = \frac{c^3 \sinh^3 \xi}{2\lambda^2 f_\xi}\, \phi\,, \quad \frac{\partial w}{\partial I_2} = \frac{c^3 \sinh^3 \xi}{2\lambda^2 f_\xi}\, \psi\,, \quad \frac{\partial w}{\partial I_3} = \frac{\lambda^2 f_\xi}{2c^3 \sinh^3 \xi}\, p \tag{30}$$

From (20) we have on differentiation

$$\left.\begin{aligned}
\frac{dI_1}{d\xi} &= \frac{2c^2 \sinh \xi \cosh \xi}{f_\xi^2} - \frac{2c^2 \sinh^2 \xi\, f_{\xi\xi}}{f_\xi^3} + \frac{4c^2 \sinh \xi \cosh \xi}{\lambda^2} \\
\frac{dI_2}{d\xi} &= \frac{4c^4 \sinh^3 \xi \cosh \xi}{\lambda^4} + \frac{2c^4}{\lambda^2}\left\{ \frac{4 \sinh^3 \xi \cosh \xi}{f_\xi^2} - \frac{2 \sinh^4 \xi\, f_{\xi\xi}}{f_\xi^3} \right\} \\
\frac{dI_3}{d\xi} &= \frac{c^6}{\lambda^4}\left[\frac{6 \sinh^5 \xi \cosh \xi}{f_\xi^2} - \frac{2 \sinh^6 \xi\, f_\xi}{f_\xi^3} \right]
\end{aligned}\right\} \tag{31}$$

Putting (30) and (31) in (29) we have

$$\frac{dw}{d\xi} = \frac{c^3 \sinh^3 \xi}{\lambda^2 f_\xi}\Bigg[\left(\frac{c^2 \sinh \xi \cosh \xi}{f_\xi^2} - \frac{c^2 \sinh^2 \xi\, f_{\xi\xi}}{f_\xi^3} + \frac{2\, c^2 \sinh \xi \cosh \xi}{\lambda^2} \right)\phi + $$
$$+ \left(\frac{2c^4 \cosh \xi \sinh^3 \xi}{\lambda^4} + \frac{4c^4 \cosh \xi \sinh^3 \xi}{\lambda^2 f_\xi^2} - \frac{2c^4 \sinh^4 \xi\, f_{\xi\xi}}{\lambda^2 f_\xi^3} \right)\psi +$$
$$+ \left(\frac{3 \cosh \xi}{\sinh \xi} - \frac{f_{\xi\xi}}{f_\xi} \right) p \Bigg]$$

$$\frac{dw}{d\xi} = \frac{c^3}{\lambda^2}\frac{\partial}{\partial \xi}\left(\frac{\sinh^3 \xi\, \sigma_{11}}{f_\xi} \right) \tag{32}$$

(32) on integration gives

$$\sigma_{11} = \frac{\lambda^2 f_\xi\,(w + w_0)}{c^3 \sin h^3 \xi} \tag{33}$$

Where w_0 is an integration constant, using (20), (33) may be written as

$$\sigma_{11} = \frac{w + w_0}{\sqrt{I_3}} \tag{34}$$

Equating the two values of σ_{11} from (23) and (34) and using (6) and 20 we have

$$f_\xi = \sqrt{\frac{2 \sinh^2 \xi}{w + w_0}\left(\frac{\partial w}{\partial I_1} + \frac{2 \sinh^2 \xi}{A^2}\frac{\partial w}{\partial I_2} + \frac{\sinh^4 \xi}{A^4}\frac{\partial w}{\partial I_3} \right)} \tag{35}$$

(35) on integration gives

$$f(\xi) = \int \frac{2 \sinh^2 \xi}{w + w_0}\left(\frac{\partial w}{\partial I_1} + \frac{2 \sinh^2 \xi}{A^2}\frac{\partial w}{\partial I_2} + \frac{\sinh^4 \xi}{A^4}\frac{\partial w}{\partial I_3} \right) d\,\xi + K \tag{36}$$

Where K is an integration constant

BOUNDARY CONDITIONS

If the inner boundary of the shell $\xi = \xi_1$ is free from tractions we must have $\sigma_{11} = 0$ when $\xi = \xi_1$ which on substitution in (34) gives

$$0 = (\sigma_{11})_{\xi=\xi_1} = \frac{w(\xi_1) + w_0}{\sqrt{I_{3(\xi_1)}}} \tag{37}$$

(37) gives

$$w_0 = -w(\xi_1) \tag{38}$$

From (38) and (34) we have

$$\sigma_{11} = \frac{w(\xi) - w(\xi_1)}{\sqrt{I_3}} \tag{39}$$

on the outer surface of the shell ($\xi = \xi_2$) we have to apply a radial force R_1 given by

$$R_1 = \sigma_{11}(\xi_2) = \frac{w(\xi_2) - w(\xi_1)}{\sqrt{I_{3(\xi_2)}}} \tag{40}$$

The resultant force F_1 and the couple M_1 acting on the edge ($\eta = \alpha$) per unit area between ϕ and $\phi + d\phi$ are given by

$$F_1 = \sin\alpha \int_{\xi_1}^{\xi_2} \sigma_{22}\,(c \sinh \xi)^2\, d\xi \tag{41}$$

$$M_1 = \sin\alpha \int_{\xi_1}^{\xi_2} \sigma_{22}\, c^2 \sinh^2 \xi \;(c \cosh \xi)\; d\xi \tag{42}$$

From (26) we may write

$$\sigma_{22} = \sigma_{11} + \frac{1}{2} \tanh \xi \frac{\partial \sigma_{11}}{\partial \xi} \tag{43}$$

(43) and (34) give

$$\sigma_{22} = \sigma_{33} = \frac{w + w_0}{\sqrt{I_3}} + \frac{1}{2} \tanh \xi \frac{\partial}{\partial \xi}\left(\frac{w + w_0}{\sqrt{I_3}}\right) \tag{44}$$

(44) and (41) give.

$$2F_1 = c^2 \sin\alpha \left[\frac{w(\xi_2) - w(\xi_1)}{\sqrt{I_{3(\xi_2)}}} \sinh^2 \xi_2 - \int_{\xi_1}^{\xi_2} \frac{w(\xi) - w(\xi_1)}{\sqrt{I_3}} \tanh^2 \xi \, d\xi \right] \tag{45}$$

(44) and (42) give

$$2M_1 = c^3 \sin\alpha \left[\frac{w(\xi_2) - w(\xi_1)}{\sqrt{I_{3(\xi_2)}}} \sinh^3 \xi_2 - \int_{\xi_1}^{\xi_2} \frac{w(\xi) - w(\xi_1)}{\sqrt{I_3}} \sinh^2 \xi \cosh \xi \, d\xi \right] \tag{46}$$

Thus to bend an isotropic compressible circular block into a part of an ellipsoidal shell, we require a resultant force F_1 and a couple of moment M_1 on the edge and a radial force R_1 on the outer surface of the ellipsoidal shell

PARTICULAR CASES

Case I. Bending of incompressible circular block in to an ellipsoidal shell.

Putting $I_3 = 1$, (45) and (46) give

$$2F_1 = c^3 \sin\alpha \left[\left\{ w(\xi_2) - w(\xi_1) \right\} \sinh^2 \xi_2 \right] \tag{47}$$

$$2\,M_1 = c^3 \sin\alpha \left[\left\{ w(\xi_2) - w(\xi_1) \right\} \sinh^3 \xi_2 - \int_{\xi_1}^{\xi_2} \left\{ w(\xi_2) - w(\xi_1) \right\} \sinh^2 \xi \cosh \xi d\xi \right] \quad (48)$$

Case II. Bending of an isotropic compressible circular block into a spherical shell.

If we put $c \cosh \xi_i = c \sinh \xi_i$ in (11), we get the case of a circular block bent into a spherical shell, so that $\xi \to \infty$, $c \to \infty$ and $c \cosh \xi_i\ c \sinh \xi \to r$ and consequently the orthogonal curvilinear coordinates (ξ, η, ϕ) are replaced by the spherical polar coordinates (r, η, ϕ). Then (45) and (46) reduce to

$$2\,F_1 = \frac{r_2^2 \sin\alpha}{\sqrt{I_3(r_2)}} \left[w(r_2) - w(r_1) \right] \quad (49)$$

$$2\,M_1 = \sin\alpha \left[\frac{r_2^3 \{ w(r_2) - w(r_1) \}}{\sqrt{I_3(r_2)}} - \int_{r_1}^{r_2} \frac{r^2 \{ w(r) - w(r_1) \}}{\sqrt{I_3}}\, dr \right] \quad (50)$$

Case III. Bending of incompressible circular block into spherical shell

Putting $I_3 = 1$, (49) and (50) give

$$2\,F_1 = r_2^2 \sin\alpha \left[w(r_2) - w(r_1) \right] \quad (51)$$

$$2\,M_1 = \left[w(r)\, r_2^3 - w(r_1)\, \frac{2 r_2^3 + r_1^3}{3} + \int_{r_1}^{r_2} r^2 w\, dr \right] \sin a \quad (52)$$

REFERENCES

1. RIVLIN, R.S., *Phil. Trans. Roy. Soc. (London)*, **A 240** (1948), 459-490, 491, 508, 509—525.
2. GREEN, A.E. & ZERNA, W., 'Theoretical Elasticity, (Oxford Press, Oxford), 1954.
3. GREEN, A.E. & ADKINS, J.E., 'Large Elastic Deformations and Continue Mechanics' (Oxford Press, Oxford), 1960.
4. SETH, B.R., *ZAMM*, **37** (1957), 393—398.
5. SETH, B.R., *Proc. Ind. Acad. Sci.*, **A 30** (1957), 105—112.
6. LAKSHMINARAYANA, G., *Proc. Nat. Acad. Sci. (India)*, **29** (1960), 297—306'
7. LAKSHMINARAYANA, G., *ZAMM*, **43**, (1963), 563—64.
8. SETH, B.R., *Phil. Trans. Roy. Soc. (London)*, **A 234** (1935), 261—64.