THE BAYESIAN APPROACH TO THE ECONOMICS OF ONE-TIME INVENTORY POLICY UNDER UNCERTAINTY

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This paper deals with the one-time buy inventory problem of such products for which there is limited information available for demand prediction. The unknown mean demand is treated as random variable whose prior density is defined by a Beta density function. The Baye's rule is then used to redefine the economics of the inventory model under uncertainty about the mean demand. Assuming that no cost is incurred on collection of data, the expected cost of ignorance becomes the limiting value of the expected savings which accrue due to the increase in knowledge about the true mean demand.

The News Vendor's problem[1] concerning stock is well known, that an item which can be stocked at the beginning of a time period only, is considered for the determination of its inventory level. If this level proves to be insufficient for the demands which occur during the period, no additional stock can be procured until the beginning of the next subsequent stocking period. The left over stock at the end of a period becomes useless and obsolete such that it can not be used to satisfy the subsequent demands. While dealing with such problems, it is usually assumed that the customer's demand to be a random variable with a known distribution to follow. However, when one is dealing with a new product and for which there is limited information or experience available for demand prediction, one is faced with an uncertainty about the demand behaviour. Also, with the change in the market conditions, one might observe a variation in the mean demand during each inventory period.

In this paper, the mean demand is, therefore, treated here as a random variable whose density function defines its characteristic behaviour. The Baye's rule is then used to redefine the economics of the inventory model and the optimal inventory level is obtained under uncertainty about the mean demand. It has been shown that for a given prior knowledge about the mean demand, the expected cost of ignorance becomes the limiting value of the expected savings which accrue due to the increase in knowledge about the true mean demand.

SINGLE PERIOD MODEL AND COST STRUCTURE

Assume that the stock is maintained for a single period of unit length, such that the unsatisfied demands are not back ordered to a future period. However, the demands not met out of inventory are satisfied in a non-routine ways at additional costs. The cost structure is then assumed to be such that the cost is proportional to the stock held at the beginning of the period and to a cost of failing to meet demands which is proportional to the amount of the shortage occuring during the period. The expected cost of holding a stock of level I is given by[2]

$$E(I) = C\,I + C_p \sum_{x=I}^{\infty} (x - I)\, f(x) \tag{1}$$

where

$f(x)$ = the discrete demand distribution

x = the demand during the period

C = unit cost of holding a stock at the beginning of the period

C_p = unit cost of shortage or penury.

and if I_o be the optimal level of inventory, one should have

$$\sum_{x=0}^{I_o} f(x) = \frac{C_p - C}{C_p} \tag{2}$$

On substituting the optimal value I_o from (2) to (1), one obtains the expected cost of an optimal policy as

$$E(I_o) = C_p \sum_{x=I_o}^{\infty} x f(x) \tag{3}$$

Now, that the demand distribution is known with uncertainty about the mean demand, let $f(x)$ be distributed as Poisson with the mean demand λ. The uncertainty about λ is expressed by regarding it as a random variable governed by a Beta—*prior density,* viz.

$$g(\lambda) = \frac{\lambda^{\nu_1-1}(1-\lambda)^{\nu_2-1}}{B(\nu_1, \nu_2)} \tag{4}$$

where

$$\nu_1, \nu_2, > 0 \; ; \; 0 < \lambda < 1.$$

and $B(\nu_1, \nu_2)$ is a complete Beta function.

Then, according to the author[3] the unconditional demand distribution is given by

$$f(x) = \frac{B(x+\nu_1, \nu_2)}{B(\nu_1, \nu_2)\, x\,!}\; {}_1F_1(x+\nu_1 \,;\, x+\nu_1+\nu_2 \;;\, -1) \tag{5}$$

with mean demand $= \dfrac{\nu_1}{\nu_1+\nu_2}$

and ${}_1F_1(. \;;\; . \;;\; -1)$ is a confluent hypergeometric function[4] Having known the prior density for λ, one obtains the optimal level of the inventory I_o from (2) and (5) by solving the following equation.

$$\sum_{x=0}^{I_o} \frac{B(x+\nu_1, \nu_2)}{B(\nu_1, \nu_2)\, x\,!}\; {}_1F_1(x+\nu_1 \,;\, x+\nu_1+\nu_2 \,;\, -1) = \frac{C_p - C}{C_p} \tag{6}$$

Assuming S_i be the sum of the first $(i+1)$ terms of the L.H.S. of the above equation, a recurrence formula is derived at Appendix as follows

$$S_i = S_{i-1} + \frac{\nu_1+\nu_2+i-1}{i}(S_{i-1} - S_{i-2}) - \frac{\nu_1+i-2}{i\,(i-1)}(S_{i-2} - S_{i-3}) \tag{7}$$

when $i \geqslant 2$. For other values of i, the eqns. (α) and (β) of the Appendix are used. The optimal value I_o is obtained by solving the (6) and (7) such that $i = I_o$ when

$$S_i = \frac{C_p - C}{C_p} \tag{8}$$

IGNORANCE COST

The expected cost for an optimal inventory I_o based on the prior knowledge of λ, is obtained from (3) and (5) as

$$E_{pr}(I_o) = C_p \sum_{x=I_o}^{\infty} \frac{B(x+\nu_1, \nu_2)}{B(\nu_1, \nu_2)\,.\,(x-1)\,!}\; {}_1F_1(x+\nu_1;\; x+\nu_1+\nu_2;\; -1)$$

Using the recurrence formula (7), the above equation reduces to

$$E_{pr}(I_o) = C_p \left[\frac{\nu_1}{\nu_1+\nu_2} - I_o \frac{C_p - C}{C_p} + \sum_{i=0}^{I_o-1} S_i \right] \tag{9}$$

Now, that the true mean demand is not known, the optimal stock holding based on the prior knownledge about λ, results in the extra cost which measures the expected cost of ignorance about the true mean demand, i.e.

$$\text{Expected Ignorance Cost} = E_{pr}(I_o) - E(\hat{I}_o/\hat{\lambda}) \tag{10}$$

where

$\hat{I}_o$ = the optimal stock level when the true mean demand $\hat{\lambda}$ is known.

THE ECONOMICS OF SAMPLE INFORMATION

Due to the limited knowledge about the true mean demand, the evaluation of the expected ignorance cost is just not possible. As such, one is, therefore, required to improve one's knowledge either through some market studies which would be relevant or through operating for a period which results in the observation of demands. The data so obtained may be relevant to the period one is planning for. The Baye's rule is then used to revise one's opinion about the demand which may arise in future periods. Given such information in the form of sample demand data, the prior density is transformed into the posterior density for λ which in turn leads to the posterior distribution of demand and the optimal inventory level is then computed.

Let the sample information about the demand is obtained for r—successive independent periods each of unit length. The posterior density of λ as shown by the author[3] is given as follows:

$$g^*(\lambda) = \frac{e^{-r\lambda} \cdot \lambda^{A_r + \nu_1 - 1} \cdot (1-\lambda)^{\nu_2 - 1}}{B(A_r + \nu_1, \nu_2)\ {}_1F_1(A_r + \nu_1 ; A_r + \nu_1 + \nu_2 ; -r)} \quad ; \quad A_r = \sum_{i=1}^{r} x_i \tag{11}$$

and the posterior demand distribution given $x_1, x_2, \ldots x_r$ number of demands observed during the past r-independent successive periods each of unit length, is obtained as

$$f(Y = y/X = x_1, x_2, \ldots, x_r)$$

$$= \frac{B(y + A_r + \nu_1, \nu_2)\ {}_1F_1(y + A_r + \nu_1 ; y + \nu_1 + \nu_2 + A_r ; -r+1)}{B(A_r + \nu_1, \nu_2), y!\ {}_1F_1(A_r + \nu_1 ; A_r + \nu_1 + \nu_2 ; -r)} \tag{12}$$

The optimal inventory level I_o'' for this posterior demand distribution is obtained through (2) by solving the following equation for I_o''.

$$\sum_{y=0}^{I_o''} \frac{B(y + A_r + \nu_1, \nu_2)}{B(A_r + \nu_1, \nu_2) \cdot y!} \ \frac{{}_1F_1 \ (y \times A_r + \nu_1 ; y + A_r + \nu_1 + \nu_2 ; \ \overline{r+1})}{{}_1F_1 \ (A_r + \nu_1 ; A_r + \nu_1 + \nu_2 ; -r]} = \frac{C_p - C}{C_p} \tag{13}$$

The optimal value I_o'' is however obtained through the recurrence formula given at *Appendix* on the lines as given in (8). And the subsequent expected cost of holding at optimal policy based on the posterior knowledge about λ, is obtained through (3) and (13) as

$$E_{po}(I_o'') = C_p \left[\frac{A_r + \nu_1}{A_r + \nu_1 + \nu_2} \cdot \frac{{}_1F_1(A_r + \nu_1 + 1 ; A_r + \nu_1 + \nu_2 + 1 ; -r)}{{}_1F_1(A_r + \nu_1 ; A_r + \nu_1 + \nu_2 ; -r)} - \right.$$

$$\left. - I_o'' \frac{C_p - C}{C_p} + \sum_{i=0}^{I_o''-1} S_i'' \right] \tag{14}$$

where S_i'' = the sum of the first $(i+1)$ terms of the L.H.S. of eqn. (13).

The economy of holding the inventory at I_o'' level is, therefore, measured as

expected savings $= E_{pr}(I_o) - E_{po}(I_o'')$

$$= C_p \left[\frac{\nu_1}{\nu_1 + \nu_2} - \frac{A_r + \nu_1}{A_r + \nu_1 + \nu_2} \cdot \frac{{}_1F_1(A_r + \nu + 1 ; A_r + \nu_1 + \nu_2 + 1 ; -r)}{{}_1F_1(A_r + \nu_1 ; A_r + \nu_1 + \nu_2 ; -r)} + \right.$$

$$\left. + \frac{C_p - C}{C_p}(I_o'' - I_o) - \sum_{i=0}^{I''_o - 1} S_i'' + \sum_{i=0}^{I_o - 1} S_i \right] \tag{15}$$

THE LIMITING VALUE OF EXPECTED SAVINGS

As the number of periods of observation r is increased, the knowledge about the true mean demand $\hat{\lambda}$ increases and the limiting value of the expected savings (15) as r tends to infinity, leads to the following according to the author[3] as

expected savings at $r \uparrow \infty$

$$= C_p \left[\frac{\nu_1}{\nu_1 + \nu_2} - \hat{\lambda} + \frac{C_p - C}{C_p} (I_o' - I_o) - \sum_{i=0}^{I_o'-1} S_i' + \sum_{i=0}^{I_o'-1} S_i \right] \tag{16}$$

where

S'_i = the sum of the first $(i + 1)$ terms of the Poisson distribution with $\hat{\lambda}$ as the mean.

I'_o = optimal level of inventory based on the knowledge as $r \uparrow \infty$.

In other words, the expected saving at $r \uparrow \infty$, referring to (9) leads to

$$= E_{pr} (I_o) - E (I_o'/\hat{\lambda}) \tag{17}$$

Now, that the limiting value of the mean demand tends to $\hat{\lambda}$ (the true mean demand) as $r \uparrow \infty$, the optimal inventory level I_o' will also tend to $\hat{I}_o$.

Thus

$$E_{pr} (I_o) - E (I_o'/\hat{\lambda}) = E_{pr} (I_o) - E (\hat{I}_o/\hat{\lambda}) \tag{18}$$

= the expected cost of ignorance (refer eqn. 10).

EXAMPLE

Airforce is required to make its buying decision several years prior to the delivery of the first end item of a system, and if the design of the system changes before its phase-out, the spare parts are procured at premium cost only. Therefore, as normally happens, let the total requirement of spares for the operational period of a system be made at the time when the buying decision is taken. However, all the spares delivered and on the pipeline become obsolete once the phase-out of the system has taken place. Under such condition, the provisioning of spares is made normally with a unknown demand rate. The normal practice is that the demand experience on similar parts and on similar system are used by the provisioner and the spare part requirement is made. Thus, such system primarily involves the expenses associated with holding an item in the inventory.

Assuming, that the unkown demand rate (λ) be governed by a Beta-Prior density whose parameters, say, to be $\nu_1 = 0.5$ and $\nu_2 = 0.2$ obtained on the lines as indicated by the author[3]. It is, now, desired to find the optimal inventory of the system and its expected cost of holding and the expected savings, if any, when an item of the system is maintained for the unit (= T a/c months) period of operation under the following conditions :

(*i*) that the demand experience used on similar system are available for six successive independent sample occasions of same duration (= T a/c months) each, and the total demand for that item observed to be zero (say),

or (*ii*) that under similar condition of experience as cited under (i) above, the total demand for that item observed to be unity (say).

Having known the values of ν_1 and ν_2, the euqation (7) gives the optimal value for a given prior as

$$S_4 = 0.9980615$$

Therefore, if one is satisfied of having a risk at 0.2% that the demand may exceed the optimal value (= 4) in a future period of T a/c months, the expected cost of holding the inventory at this optimal value (= 4) based on prior knowledge about λ is obtained from (9) as

$$E_{pr}\ (I_o = 4) = C_p\ (0.0102901)$$

Now, under condition (ii), the optimal inventory level at 0.2% risk of being stock-out, is obtained from (13) as $I_o'' = 2$, and the expected cost of holding at this posterior optimal value is given by (14) as

$$E_{po}\ (I_v'' = 2) = C_p\ (0.0074749)$$

The expected savings of holding the stock at posterior knowledge over that of prior is now obtained from (15) as

$$\text{Expected Savings} = C_p\ (0.0028152)$$

Likewise, if the demand for a spare part observed during the six consecutive periods each of T a/c months to be unity, then repeating the process as above, the posterior optimal value is obtained as $I_o'' = 3$ at 0.2% risk of being stockout, and its expected cost of holding for an equal period (= T) is

$$E_{po}\ (I_o'' = 3) = C_p\ (0.0095364)$$

When the expected savings over prior knowledge is obtained as = C_p (0.0007537).

CONCLUSION

Whenever, one is required to take a decision on the procurement of spare parts along with the new system with a view that the quanity of spares will be sufficient to meet the requirements until the system is phased out, the present theory provides a solution for the optimal stock level to be maintained at a desired level of risk of being stock out. The uncertainty about the average demand is reduced by way of past experience. The reduction in the cost of ignorance depends on the prior information and that of the subsequent sample information on the demand behaviour and is measured as theexpected cost of savings.

A Poisson distribution with unknown λ and a known prior distribution for λ amounts to that the demand distribution is known and it would be better to treat the demand rate as a parameter to be estimated in the classical manner and then worked out the rest of problem—a common view.

Prior to the experimentation, the information available on the parameter could be summarised into a prior density function called as prior density and with the process of learning after experimentation, this density function is modified and a new demand distribution is obtained. Thus, one observes that the unconditional demand distribution does not remain the same after experimentation [refer eqns. (5) & (12)].

The conventional method of estimation does not admit the 'parameter' as a random variable and the 'Bayesian estimates' is therefore sought for[5].

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REFERENCES

1. KAUFMANN, A. & FAURE, R., 'Introduction to Operation Research' (Academic Press), 1968, p. 9.
2. MORRIS, WILLIAM, T., 'Management Science, A. Bayesian Introduction' (Prentice Hall Inc., N. Y.), 1968, p. 110.
3. BHATTACHARYA, D. D., *Def. Sci. J.*, **29** (1979), 3.
4. SLATER, L. J., 'Confluent Hypergeometric Functions' (Camb. Uni.), 1960, p. 34.
5. BRUNK, H. D., 'An Introduction to the Mathematical Statistics' (Xerox College Publishing, Lexington, Toronto), 1975, p. 229.

Appendix

RECURRENCE FORMULA

The value of I for which the following equation is satisfied, gives the optimal inventory level for one-time buy policy:

$$\sum_{y=0}^{I} \frac{B(y+A_r+\nu_1, \nu_2)}{B(A_r+\nu_1, \nu_2) \cdot y!} \quad \frac{{}_1F_1(y+A_r+\nu_1 ; y+A_r+\nu_1+\nu_2 ; -\overline{r+1})}{{}_1F_1(A_r+\nu_1 ; A_r+\nu_1+\nu_2 ; -r)} = \frac{C_p - C}{C_p}.$$

In order to solve the above equation, for I, a recurrence formula for the L.H.S. is developed as under :

$$\text{Let } S_i = \sum_{y=0}^{i} \frac{B(y+A_r+\nu_1, \nu_2)}{B(A_r+\nu_1, \nu_2) \cdot y!} \quad \frac{{}_1F_1(y+A_r+\nu_1 ; y+A_r+\nu_1+\nu_2 ; -\overline{r+1})}{{}_1F_1(A_r+\nu_1 ; A_r+\nu_1+\nu_2 ; -r)}$$

$$= \sum_{y=0}^{i} \frac{(A_r+\nu_1)\hat{y}}{(A_r+\nu_1+\nu_2)_y \cdot y!} \quad \frac{{}_1F_1(y+A_r+\nu_1 ; y+A_r+\nu_1+\nu_2 ; -\overline{r+1})}{{}_1F_1(A_r+\nu_1 ; A_r+\nu_1+\nu_2 ; -r)}$$

where

$$(A_r+\nu_1)_y = (A_r+\nu_1)\ (A_r+\nu_1+1)\ldots\ldots(A_r+\nu_1+y-1).$$

Using the Kummer's first solution[4], one finds

$$S_0 = e^{-1} \cdot \frac{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r ; r+1)}{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r ; r)} \tag{α}$$

and

$$S_1 = S_0 + \frac{\nu_1+A_r}{\nu_1+\nu_2+A_r} \cdot e^{-1} \cdot \frac{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r+1 ; r+1)}{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r ; r)} \tag{β}$$

and so on.

accordingly

$$S_i = S_{i-1} + \frac{(A_r+\nu_1)_i}{(A_r+\nu_1+\nu_2)_i} \cdot \frac{e^{-1}}{i!} \quad \frac{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r+i ; r+1)}{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r ; -r)}$$

Applying, the recurrence relation[4] for ${}_1F_1(\ldots)$ one obtains

$$S_i = S_{i-1} + \frac{S_{i-1} - S_{i-2}}{i(r+1)} \left(\nu_1+\nu_2+A_r+i+r-1 \right) -$$

$$- \frac{(\nu_1+A_r)_{i-1}}{(\nu_1+\nu_2+A_r)_{i-2}} \cdot \frac{e^{-1}}{i!} \quad \frac{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r+i-2 ; r+1)}{{}_1F_1(\nu_2 ; \nu_1+\nu_2+A_r ; r) \cdot (r+1)}$$

$$= S_{i-1} + \frac{\nu_1+\nu_2+A_r \times i+r-1}{i(r+1)} \left(S_{i-1} - S_{i-2} \right) -$$

$$\frac{\nu_1+A_r+i-2}{i(i-1)(r+1)} \left(S_{i-2} - S_{i-3} \right). \tag{γ}$$

where

$$i \geqslant 2 \ \&\ S_{-ire} = 0 .$$