SOME PHYSICAL PROPERTIES OF PLANE DIABATIC GAS FLOWS

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Considering an orthogonal net in a plane of flow related to the streamlines and their orthogonal trajectories, general physical and geometrical properties of steady diabatic flow are examined. This exploration answers how far the physical laws are governed by the geometric descriptions of the diabatic flow. Radial diabatic flow fields are characterized, from which the flow described by Pai[6] can be deduced.

Quite a good number of problems in aerodynamics and meteorology break through the concept of adiabatic, as the role of heat content cannot be neglected. In general, the heat is either generated or emitted or absorbed during the dynamics state of gas. Considering these facts for the first time, Champman and Jouguet, around 1900 during early part of combustion phenomena formulated correctly some fluid dynamical problems involving changes in total temperature. Kiebel's[1] works classified viscous, compressible flow into a number of dynamically permissible types with application to meteorology. Hicks[2] Donald Chereweth[3] have studied the physical properties of diabatic flow fields. The role of heat content and the non-linear character of the partial differential equations governing diabatic flow fields have presented a series of stringent mathematical complexities. Exploiting the geometric techniques Purushotham and Madhusudan[4], Madhusudan[5] have studied the diabatic flows and have obtained possible flow fields in particular cases.

In the present investigation defining plane orthogonal net related to the streamlines and their orthogonal trajectories, various physical and geometrical properties of diabatic flows are studied in more general way, from which analytical flows can be obtained if the distribution of heat sources and sinks are advanced as depicted in Fig. 1.

Considering the radial distribution of heat content, radial flows are studied in which isovels touch the streamlines and a possible flow is obtained from which the flow discussed by Pai[6] can be deduced as a special case and Chapligan's gas flow fields can be determined.

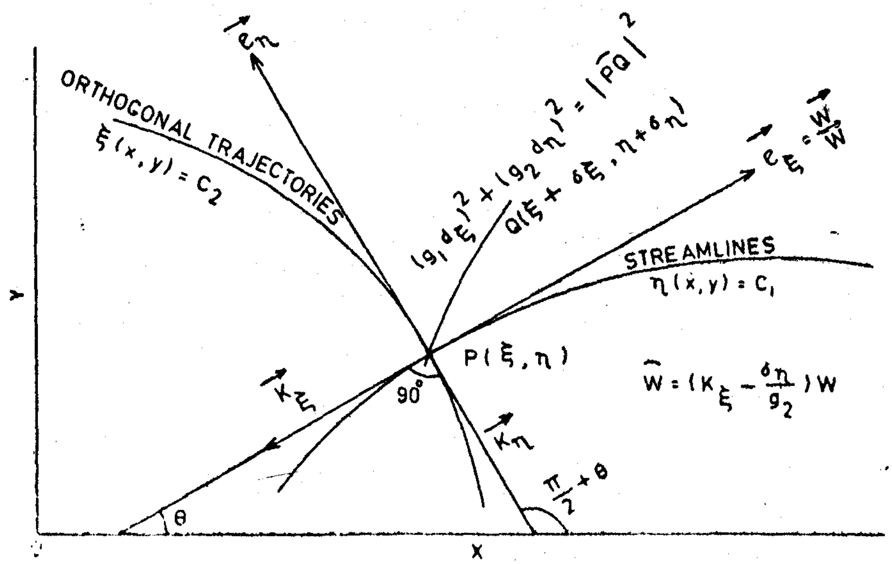

Fig. 1—Streamline geometry—flow representation

FUNDAMENTAL EQUATIONS

The fundamental equations governing the diabatic steady gas flows in absence of viscosity and extraneous forces are given below in the usual notation in Crocco's velocity vector field[2].

$$div \left\{ \vec{W} (1 - W^2)^{\frac{1}{\gamma - 1}} \right\} = q \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2 \right) \left(1 - W^2 \right)^{\frac{2-\gamma}{\gamma - 1}} \tag{1}$$

$$\nabla \log p_t = \frac{2\gamma}{(\gamma - 1)(1 - W^2)} \left(\vec{W} \times curl \vec{W} - q\vec{W} \right) \tag{2}$$

$$p_t = p (1 - W^2) \tag{3}$$

$$T_t = T (1 - W^2)^{-1} \tag{4}$$

where $\vec{W}$, q, γ, p_t, $C_p T_t$, T, S and W are the reduced velocity vector, the heat content, the adiabatic exponent, the total pressure, the stagnation enthalpy, the temperature, the specific entropy and the magnitude of the Crocco's velocity vector respectively.

In addition to these we write the Crocco's vorticity relation for adiabatic flow as

$$V_t \vec{W} \times curl \left(V_t \vec{W} \right) = C_p \nabla T_t - T \nabla S \tag{5}$$

This shall be used to study the adiabatic flows. Operating *curl* on (2) and eliminating the total pressure p_t we obtain

$$curl \left(\frac{\vec{W} \times curl \vec{W} - q\vec{W}}{1 - W^2} \right) = 0 \tag{6}$$

These constitute the integrability for all steady diabatic inviscid gas flows, from which we can deduce the conditions derived by Nemenyi and Prim[7] and others[4,5] governing adiabatic gas flows.

PHYSICAL AND GEOMETRIC PROPERTIES OF FLOWS

Considering a family of orthogonal curves of congruences η (x, y)-constant and t ξ (x,y)-constant as streamlines and their orthogonal trajectories, in a plane of flow and θ as the inclination of streamline with fixed direction (x-axis), we have the following geometric results.

$$g_1^{-1} \partial_\xi \theta = - g_2^{-1}, \quad \partial_\eta \log g_1 = k_\xi \tag{7a}$$

$$g_2^{-1} \partial_\eta \theta = g_1^{-1} \partial_\xi \log g_2 = k_\eta \tag{7b}$$

$$\partial_\xi \left(g_1^{-1} \partial_\xi g_2 \right) + \partial_\eta \left(g_2^{-1} \partial_\eta g_1 \right) = 0 \tag{8}$$

where (∂_ξ , ∂_η) and (g_1 , g_2) are the partial differential operators and the first fundamental metric coefficients.

The differential relation (8) is due to Gauss[8].

From the above geometric considerations the velocity vector field $\vec{W}$ can be expressed as $\vec{W} = \vec{e\xi}\, W$, where $\vec{e\xi}$ and W are the unit vector in ξ— increasing directions and the magnitude of the velocity vector field respectively.

Using this and (7*b*) in (1), we decompose

$$\frac{1 + \lambda W^2}{W(W^2 - 1)} \left(q + g_1^{-1} \partial_\xi W \right) = k = g_1^{-1} \partial_\xi \log g_2 = g_2^{-1} \partial_\eta \theta \tag{9}$$

It is readily seen from (9) that the flow is expansive in the regions, restricted by the set of conditions

$C_p > C_v$, $W - W_0 \gtrless \int_{\xi_0}^{\xi} -(g_1\, q)\, d\xi$ and $W \gtrless 1$, where W_0 is defined at points $\xi = \xi_0$ in a region of flow.

The integral is well defined, if the distribution of heat sources and sinks are advanced. The only integral condition is reversed for compressive flow.

The above stated analysis also hold for Chaplygins gas flows. Further if isovels touch the streamlines and $W \gtrless 1$, the flow is either expansive or compressive according to the presence of heat sources or sinks respectively.

The expansive and compressive nature of the flow is associated with convexity and concavity character of orthogonal trajectories of the streamlines.

If the cross-section of the stream tube is stationary and the heat sources and sinks are given then (9) simplifies to

$$g_1\, q = \partial_\xi\, W \tag{10}$$

So, from (10) we infer that the velocity increases or decreases along a streamline in the presence of heat sources or sinks respectively. The flow can be evaluated from (10), if the distribution of heat and the geometry of streamlines are known.

The above results can be suitably modified for adiabatic phenomenon.

Forming the scalar product of (2) by $\vec{e}_\xi$, we obtain

$$g_1^{-1}\, \partial_\xi \log p_t = \frac{2\gamma\, qW}{(1-\gamma)(1-W^2)} \tag{11}$$

So the total pressure remains uniform along an individual streamline for all adiabatic gas flows ($q=0$).

Further from (11), the total pressure along a streamline can be determined completely in the following form, if the flow pattern distribution of heat and the flows are defined.

$$p_t - p_t^{\circ} = \exp \int_{\xi_0}^{\xi} 2\gamma\, W\, q\, g_1 \left(1-\gamma\right)^{-1} \left(1 - W^2\right)^{-1} d\xi \tag{12}$$

Thus the geometry and the physics of the flows describe the flow structure.

The relation (11) can also be written as

$$\frac{p_t - p_t^{\circ}}{\xi - \xi_0} = \underset{\xi \to \xi_0}{at} \exp\left(\frac{2\gamma\, q\, W\, g_1}{(\gamma-1)(1-W^2)}\right) \tag{13}$$

From (13) it is concluded that the cavities do not exist for diabatic flows.

The momentum relation along an orthogonal trajectory yields

$$g_2^{-1}\, \partial_\eta \log p_t = \frac{-2\gamma W}{(\gamma-1)(1-W^2)}\, \bar{\omega} \tag{14}$$

So, the potential flows exist only when the isobars touch orthogonal trajectories.

The total fluid pressure increases along an orthogonal trajectory if $\gamma > 0$, and $(1-\gamma) > 0$, $(1 - W^2) > 0$, the pressure is decreasing function along orthogonal trajectories, if $0 < W < 1$, $\gamma < 0$ and $\bar{\omega} > 0$.

The integrability condition (6) simplifies to

$$g_2^{-1}\, \partial_\eta A_\xi - g_1^{-1}\, \partial_\xi A_\eta = A_\eta\, K + A_\xi\, K_\xi$$

$$\text{where } A_\xi = qW\left(W^2 - 1\right)^{-1} \text{ and } A_\eta = \frac{W(1-W^2)}{g_1\, g_2}\, \partial_\eta (g_1 W) \tag{15}$$

The above relations are true for diabatic flows in general and particular flows can be obtained assigning various geometries to the streamlines.

For all adiabatic flows, the above equations can suitably be modified and the Crocco's vorticity relation (5) has to be read as under

$$(W V_t)^2 \partial_\eta (g_1 W V_t) = C_p \partial_\eta T_t - T \partial_\eta S \tag{16}$$

RADIAL FLOW FIELDS

In this section it is proposed to find analytical radial diabatic flows, in which the flow fields are functions of the radial distance, which include the flows discussed by Pai[6].

Following Pai[6], the Crocco's velocity vector field $\vec{W}$ can be expressed as

$$\vec{W} = W (r) \vec{i_r} \tag{17}$$

where $\vec{i_r}$ is the unit radial vector.

The integrability condition (6) is satisfied for the distribution of heat sources or sinks; since the heat content is independent of θ.

The principle of conservation of mass expressed by (1) can be read as

$$\frac{d}{dr} \log \left[r W \left(1 - W^2 \right)^{\frac{1}{\gamma - 1}} \right] = \frac{q \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2 \right)}{W (1 - W^2)} \tag{18}$$

The analytical flows can be obtained from (18), if the distribution of heat sources and sinks are advanced.

A possible flow of (18) is

$$W (1 - W^2)^{1/\gamma - 1} = \frac{\lambda}{r} e^{\int f (r) \, dr} \tag{19}$$

$$q = \frac{W (1 - W^2)}{1 + \frac{\gamma + 1}{\gamma - 1} W^2} f (r) \tag{20}$$

where $f(r)$ is an arbitrary function of r.

If $f(r) = k$ say, the general flows are

$$W \left(1 - W^2 \right)^{1/\gamma - 1} = \frac{\lambda}{r} e^{kr} \tag{21}$$

$$q = \frac{kW (1 - W^2)}{1 + \frac{\gamma + 1}{\gamma - 1} W^2} \tag{22}$$

From these we can obtain the flows discussed by Pai[6], by selecting the parameters λ and k properly, from which Chaplygin's gas flows can be deduced as a special case.

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