LIFE TEST BASED ON PROGRESSIVELY GROUP-CENSORED SAMPLES FROM EXPONENTIAL DISTRIBUTION WITH PERIODIC CHANGE IN FAILURE RATE

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A life test experiment based on progressively group-censored sample with periodic change in the failure rate of an exponential distribution is considered. Estimates of the two failure rates under two different conditions of usage of an item together with their asymptotic standard errors are obtained by the method of maximum likelihood. A numerical example is given using the data available in the form of grouped observations under two conditions of usage during alternate time intervals of fixed lengths T_1 and T_2.

KEY WORDS

Life test	Progressively	Parameters
Failure rate	Censored	Exponential distribution
Periodic change	Grouped observations	Estimation

In life testing, it is a common practice to terminate the experiment when a certain number of items have failed or a certain stipulated time has elapsed. Cohen[1] introduced progressively censored samples in life testing and recommended their use when at various stages of an experiment some of the surviving items are withdrawn (censored) from further observation and the experiment is continued with the remaining items. Several writers have investigated the maximum likelihood estimation of the parameters of the normal, log-normal, exponential and Weibull distributions when samples are progressively censored.

There are situations in which life-test is carried out periodically and the items that have failed during each time interval (T_{i-1}, T_i), are counted and some fixed number of items selected at random from the surviving ones is withdrawn (censored) at fixed points of censoring T_i $(i=1, 2, \ldots, k)$. This kind of experimentation stems from economic or practical considerations, where it may not be appropriate to collect the exact failure times of items on test and to continue testing until all the items have failed. Srivastava[2] has considered a life-test based on grouped observations from an exponential distribution under the assumption that the failure rate is constant though different under two different conditions of usage. A trivariate and a multivariate extensions of this life test have recently been studied by Patel & Gajjar[3].

According to Srivastava[2], we have the following scheme

Cycle	Phase	Duration of time	Failure rate
j	1	$\{(j-1)(T_1+T_2),\ (j-1)(T_1+T_2)+T_1\}$	θ_1
	2	$\{(j-1)(T_1+T_2)+T_1,\ j(T_1+T_2)\}$	θ_2

where $J = 1, 2, \ldots$

We note that each cycle is of time duration (T_1+T_2), the failure rate being θ_1 during the first phase of duration T_1 and θ_2 during the second phase of duration T_2. The maximum likelihood estimates of θ_1 and θ_2 have been obtained by Srivastava when the sample is singly or doubly truncated/censored and the sample observations are grouped in each phase of the cycle. In this paper, the maximum likelihood estimates of the parameters θ_1 and θ_2 along with their asymptotic standard errors are obtained when the sample is progressively censored. It is interesting to note that the maximum likelihood estimating equations obtained by Srivastava[2] an be deduced as particular cases of those in section under "Maximum Likelihood Estimation" of this paper.

THE MODEL AND LIKELIHOOD

Let X, the life span of an item, follow an exponential distribution with density

$$f(x, \theta) = \theta\, e^{-\theta x}, \text{ for } x > 0,\ \theta > 0, \tag{1}$$

When the item on test is subjected to conditions of usage, the density function of X is

$$f(x) = \begin{cases} f_1^{(j)}(x) = \theta_1\, e^{-\theta_1 x + (j-1)(\theta_1 - \theta_2) T_2}, \\ \qquad \text{for } \{ (j-1)(T_1 + T_2) < x \leqslant (j-1)(T_1 + T_2) + T_1 \\ f_2^{(j)}(x) = \theta_2\, e^{-\theta_2 x - j(\theta_1 - \theta_2) T_1}, \\ \qquad \text{for } \{ (j-1)(T_1 + T_2) + T_1 < x \leqslant j(T_1 + T_2) \} \end{cases} \tag{2}$$

where $j = 1, 2, \ldots\ldots$

The corresponding distribution function is

$$F(x) = \begin{cases} \qquad -\theta_1 x + (j-1)(\theta_1 - \theta_2) T_2, \\ \qquad \text{for } \{ (j-1)(T_1 + T_2) < x \leqslant (j-1)(T_1 + T_2) + T_1 \} \\ F_2^{(j)}(x) = 1 - e^{-\theta_2 x - j(\theta_1 - \theta_2) T_1}, \\ \qquad \text{for } \{ (j-1)(T_1 + T_2) + T_1 < x \leqslant j(T_1 + T_2) \} \end{cases} \tag{3}$$

where $j = 1, 2, \ldots\ldots\ldots$

Let N denote the total number of items put on test. We shall use the following notations :

$$i = 1, 2 \text{ and } j = 1, 2, \ldots\ldots, k$$

$$\left. \begin{aligned} n_{ij} &= \text{number of items which fail during the } i\text{-}th \text{ phase of the } j\text{-}th \text{ cycle.} \\ r_{ij} &= \text{fixed number of items which are withdrawn after the } i\text{-}th \text{ phase of the } j\text{-}th \text{ cycle.} \end{aligned} \right\} \tag{4}$$

Thus, we have

$$N = f + n_1 + n_2 + r_1 + r_2 \tag{5}$$

where

$$\left. \begin{aligned} f &= \text{number of failed items (not known) before the start of the test,} \\ n_i &= \sum_{j=1}^{k} n_{ij}\,, \quad r_i = \sum_{j=1}^{k} r_{ij} \end{aligned} \right\} \tag{6}$$

Since the test is terminated after the k-th cycle, Eqn. (5) gives

$$r_{2k} = N - f - n_1 - n_2 - r_1 - \sum_{j=1}^{k-1} r_{2j}. \tag{7}$$

Without any loss of generality, we may assume that the experiment is observed from the $(m+1)$-th cycle to the $(m+k)$-th and it is terminated after the completion of the $(m+k)$-th cycle. Thus the number f of the items failing before the beginning of the $(m+1)$-th cycle is regarded as unknown. Since the sample is censored at fixed points of time, we have the case of type I progressive group-censoring. Using the notations (4) and (6), the likelihood can be written as

$$L \propto \{ 1 - (c_1 c_2)^m \}^f \cdot \prod_{j=1}^{k} \left[\left\{ \int_{A_j}^{A_j + T_1} f_1^{(m+f)}(x)\, dx \right\}^{n_{1j}} \cdot \left\{ \int_{A_j + T_1}^{B_j} f_2^{(m+j)}(x)\, dx \right\}^{n_{2j}} \right] \cdot$$

$$\cdot \prod_{j=1}^{k} \left[\left\{ 1 - F_1^{(m+j)}(A_j + T_1) \right\}^{r_{1j}} \cdot \left\{ 1 - F_2^{(m+j)}(B_j) \right\}^{r_{2j}} \right], \tag{8}$$

where

$$\left.\begin{array}{l} A_j = (m + j - 1)(T_1 + T_2) \text{ and } B_j = (m + j)(T_1 + T_2) \text{ for } j = 1, 2, \ldots k \\ c_1 = e^{-\theta_1 T_1} \text{ and } c_2 = e^{-\theta_2 T_2} . \end{array}\right\} \tag{9}$$

MAXIMUM LIKELIHOOD ESTIMATION

Using (2), (3) and (9) the logarithm of the likelihood (8) can be written as

$$\log L = \log(\text{const}) + f \log \left\{ 1 - (c_1 c_2)^m \right\} + \log(c_1 c_2) \sum_{j=1}^{k} (m + j - 1) n_{1j} +$$

$$+ n_1 \log(1 - c_1) + n_2 \log(1 - c_2) + \sum_{j=1}^{k} r_{1j} \left\{ (m + j - 1) \log(c_1 c_2) + \right.$$

$$\left. + \log c_1 \right\} + \sum_{j=1}^{k} n_{2j} \left\{ (m + j - 1) \log(c_1 c_2) + \log c_1 \right\} + \log(c_1 c_2) \sum_{j=1}^{k} (m + j) r_{2j} \tag{10}$$

Differentiating (10) with respect to θ_1 and θ_2, in turn, we get

$$\partial \log L / \partial \theta_1 = fm \, \phi(c_1^m c_2^m) T_1 + n_1 T_1 \phi(c_1) - m n_1 T_1 -$$

$$- T_1 (m + 1)(r_1 + r_2 + n_2) - T_1 \sum_{j=1}^{k} (j - 1)(n_{1j} + n_{2j} + r_{1j} + r_{2j}) \tag{11}$$

and

$$\partial \log L / \partial \theta_2 = fm \, \phi(c_1^m c_2^m) T_2 + n_2 T_2 \phi(c_2) - m n_1 T_2 - m T_2 (r_1 + r_2 + n_2) -$$

$$- r_2 T_2 - T_2 \sum_{j=1}^{k} (j - 1)(n_{1j} + n_{2j} + r_{1j} + r_{2j}) \tag{12}$$

where

$$\phi(t) = t/(1 - t). \tag{13}$$

Equating (11) and (12) to zero, we get from the maximum likelihood equations

$$\theta_1 T_1 = \log \{ 1 + n_1 / \xi(\theta_1, \theta_2) \} \tag{14}$$

and

$$\theta_2 T_2 = \log \{ 1 + n_2 / [\xi(\theta_1, \theta_2) - n_2 - r_1] \} \tag{15}$$

where

$$\xi(\theta_1, \theta_2) = d - fm \, \phi(c_1^m c_2^m) \tag{16}$$

in which

$$d = m(n_1 + n_2) + (m + 1)(r_1 + r_2) + n_2 + \sum_{j=1}^{k} (j - 1)(n_{1j} + n_{2j} + r_{1j} + r_{2j}) .$$

Putting $r_{1j} = 0$ for $j = 1, 2, \ldots\ldots, k$ and $r_{2j} = 0$ for $j = 1, 2, \ldots, k - 1$, and $s = r_{2k} = N - n_1 - n_2 - f$ in (14), (15) and (16), we get the maximum likelihood (22) and (23) obtained by Srivastava[2].

Initial Approximate Estimates

Since (14) and (15) are transcendental equations in θ_1 and θ_2, they are not directly solvable. Therefore a suitable method of iteration like Newton's has to be employed to solve them. The initial solutions θ_1° and θ_2° with which the iteration could be started are obtained as under.

We see that n_1/n_2 is the observed ratio of the number of items failed during both the phases of the cycle. This ratio could, in the first approximation, be taken as the ratio of the expected number of failures $\theta_1 T_1/\theta_2 T_2$, since θ_1 and θ_2 are respective failure rates for durations T_1 and T_2, known in advance of the experiment. Thus we have

$$\frac{n_1}{n_2} \sim \frac{\theta_1 T_1}{\theta_2 T_2} = \frac{-\log c_1}{-\log c_2} \cdot \tag{17}$$

Now we write

$$z_i = \theta_i T_i \; (= -\log c_i) \text{ for } i = 1,\ 2 \text{ and } \rho = n_1/n_2 \cdot$$

Then from (14), (16) and (17), we get

$$\left.\begin{aligned} \xi(\theta_1, \theta_2) &= n_1 / (e^{\theta_1 T_1} - 1) \\ &= n_2 / (e^{\theta_2 T_2} - 1) + n_2 + r_1 \\ &= d - fm / (e^{m(\theta_1 T_1 + \theta_2 T_2)} - 1) \end{aligned}\right\} \tag{18}$$

and

$$z_1 \rho \doteqdot z_2 \cdot \tag{19}$$

We assume that θ_i $T_i < < 1$ $(i = 1,\ 2)$, while n_1, n_2, r_1 and r_2 are moderate. Expanding the exponential term in (18) up to the first order and noting $z_i = \theta_i\ T_i\ (i = 1, 2)$, we get

$$\begin{aligned} \xi(\theta_1, \theta_2) &\doteqdot d - f / (z_1 + z_2) \doteqdot n_1/z_1 \\ &\doteqdot n_2/z_2 + n_2 + r_1 \cdot \end{aligned} \tag{20}$$

Hence using (19) and (20), the initial approximate estimate $\theta_2{}^\circ$ of θ_2 is given by

$$\left.\begin{aligned} \theta_2{}^\circ &= T_2^{-1} \{ f \rho + n_1 (1 + \rho) \} / \{ d\rho (1 + \rho) \} \\ &= T_2^{-1} \{ f + n_2 (1 + \rho) \} / \{ (d - n_2 - r_1) (1 + \rho) \} \cdot \end{aligned}\right] \tag{21}$$

Finally the approximate estimate $\theta_1{}^\circ$ of θ_1 can be obtained from (18).

ASYMPTOTIC STANDARD ERRORS OF THE ESTIMATES

In this section, we shall obtain the asymptotic standard errors of the estimates of θ_1 and θ_2. Differentiating (11) and (12), again with respect to θ_1 and θ_2 respectively, we get

$$\partial^2 \log L / \partial\theta_1{}^2 = -fm^2 T_1{}^2 \phi (c_1{}^m c_2{}^m) / (1 - c_1{}^m c_2{}^m) - n_1 T_1{}^2 \phi (c_1) / (1 - c_1) \tag{22}$$

$$\partial^2 \log L / \partial\theta_2{}^2 = -fm^2 T_2{}^2 \phi (c_1{}^m c_2{}^m) / (1 - c_1{}^m c_2{}^m) - n_2 T_2{}^2 \phi (c_2) / (1 - c_2), \tag{23}$$

$$\partial^2 \log L / \partial\theta_1 \partial\theta_2 = -fm^2 T_1 T_2 \phi (c_1{}^m c_2{}^m) / (1 - c_1{}^m c_2{}^m) \cdot \tag{24}$$

Let $n_1{}^{(m+j)}$ and $n_2{}^{(m+j)}$ denote the numbers of items entering the first and second phase respectively in the $(m + j)$-*th* cycle. Then we have

$$\left.\begin{aligned} n_1{}^{(m+1)} &= N - f,\ n_2{}^{(m+1)} = n_1{}^{(m+1)} - n_{1\,(m+1)} - r_{1\,(m+1)} \\ n_1{}^{(m+j)} &= n_2{}^{(m+j-1)} - n_{2\,(m+j-1)} - r_{2\,(m+j-1)} \\ n_2{}^{(m+j)} &= n_1{}^{(m+j)} - n_{1\,(m+j)} - r_{1\,(m+j)} \end{aligned}\right\} \tag{25}$$

$$\text{for } j = 2,\ 3, \ldots\ldots, k$$

It is easy to verify that

$$\left.\begin{aligned} E (n_{1\,(m+j)}) &= E \{ n_1{}^{(m+j)} \} \cdot (1 - c_1), \\ E (n_{2\,(m+j)}) &= E \{ n_2{}^{(m+j)} \} \cdot (1 - c_2), \\ &\qquad \text{for } j = 1,\ 2,\ \ldots, k \\ \text{and} \qquad & \\ E (f) &= N (1 - c_1{}^m c_2{}^m) \cdot \end{aligned}\right\} \tag{26}$$

Taking expectations in (22), (23), (24) and using (26), we get

$$\left.\begin{aligned} E\left(-\partial^2 \log L / \partial \theta_1^2\right) &= N m^2 T_1^2 \phi\left(c_1^m c_2^m\right) + \\ &+ T_1^2 \phi(c_1) \sum_{j=1}^{k} E\left\{ n_1^{(m+j)} \right\}. \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} E\left(-\partial^2 \log L / \partial \theta_2^2\right) &= N m^2 T_2^2 \phi\left(c_1^m c_2^m\right) + \\ &+ T_2^2 \phi(c_2) \sum_{j=1}^{k} E\left\{ n_2^{(m+j)} \right\}. \end{aligned}\right\} \quad (28)$$

and

$$E\left(-\partial^2 \log L / \partial \theta_1 \partial \theta_2\right) = N m^2 T_1 T_2 \phi\left(c_1^m c_2^m\right). \quad (29)$$

It may be noted that the expected values of $n_1 (m+j)$ and $n_2 (m+j)$ can be calculated recursively using the relation (25) and (26).

The asymptotic standard errors of the estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ of the parameters θ_1 and θ_2 can be obtained from

$$\begin{Bmatrix} V(\hat{\theta}_1) & \operatorname{cov}(\hat{\theta}_2, \hat{\theta}_1) \\ \operatorname{cov}(\hat{\theta}_1, \hat{\theta}_2) & V(\hat{\theta}_2) \end{Bmatrix} = \begin{Bmatrix} E\left(-\partial^2 \log L/\partial \theta_1^2\right) & E\left(-\partial^2 \log L/\partial \theta_2 \partial \theta_1\right) \\ E\left(-\partial^2 \log L/\partial \theta_1 \partial \theta_2\right) & E\left(-\partial^2 \log L/\partial \theta_2^2\right) \end{Bmatrix}^{-1} \quad (30)$$

NUMERICAL EXAMPLE

To illustrate the results we assume that the experiment is observed from the second cycle and terminated after the completion of the fifth cycle, so that the number of items failed before the start of the second cycle is unknown. Here it is assumed that the observations come from an exponential distribution.

The following table gives, for the experiment, the number of tubes failed during the first 16 hours and during the remaining 8 hours of each cycle of 24 hours together with the number of tubes withdrawn from the first and the second phase of each cycle.

Thus we have

$$m = 1,\ k = 4,\ n_1 = 62,\ n_2 = 53,\ f = 87,\ r_1 = 16 \text{ and } r_2 = 82.$$

TABLE 1

FREQUENCY DISTRIBUTION OF LIVES OF VACUUM TUBES*

Duration in hours	Cycle number	Number of tubes failed during		Number of tubes withdrawn from	
		first phase n_{1j}	second phase n_{2j}	first phase r_{1j}	second phase r_{2j}
00—24	1	48	39	—	
24—48	2	25	19	5	5
48—72	3	12	10	9	8
72—96	4	17	14	1	1
96—120	5	8	10	1	68

*Total number of tubes put on test is 300.

The initial approximate estimates obtained from (21) come out to be

$$\theta^{\circ}_1 = 0.00678106 \text{ and } \theta^{\circ}_2 = 0.01159343 \ .$$

These estimates are used to solve (14) and (15) and the estimates $\hat{\theta}_1$ and $\hat{\theta}_2$, of the parameters θ_1 and θ_2, are

$$\hat{\theta}_1 = 0.0065265 \quad \text{and} \quad \hat{\theta}_2 = 0.0127326.$$

Using (30), the asymptotic standard errors of these estimates are given by

$$\begin{Bmatrix} V(\hat{\theta}_1) & \operatorname{cov}(\hat{\theta}_2, \hat{\theta}_1) \\ \operatorname{cov}(\hat{\theta}_1, \hat{\theta}_2) & V(\hat{\theta}_2) \end{Bmatrix} = \begin{Bmatrix} 1918981 \cdot 3715 & 167607 \cdot 9111 \\ 167607 \cdot 9111 & 440389 \cdot 6508 \end{Bmatrix}^{-1} = 10^{-6} \times$$

$$\begin{Bmatrix} 0 \cdot 517468 & -0 \cdot 205149 \\ -0 \cdot 205149 & 2 \cdot 348793 \end{Bmatrix}.$$

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