TRANSIENT RESPONSE OF SYMMETRICAL PARALLEL RC MULTIPLE-T NETWORKS

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Study of RC symmetrical multiple-T networks as tuning elements have been made. The behaviour of these networks to a step voltage is studied and compared with that of a series tuned LCR circuit.

Symmetrical RC multiple-T networks in parallel, as tuning elements have been studied in great detail[1,2,3]. The transient behaviour of these circuits is studied and compared with that of a series tuned LCR circuit. The conditions necessary for over damping, critical damping and underdamping are given for a multiple-T with the number n_r for resistance arms and n_c for the capacitance arms.

DERIVATION OF THE FORMULA

A multiple-T as shown in Fig. 1 is subjected to a step voltage input. The nature of the output voltage is obtained from the formula given using Stanton's[4] general values for resistances and capacitances in the twin-T networks.

The component values chosen are such that

$$\frac{R_1}{R} = \frac{C}{C_1} \tag{1}$$

$$R_1 = KR \text{ and } C_1 = C/K \tag{2}$$

$$R_2 = KR/\mu^2 (1 + K) \text{ and } C_2 = C(1 + K)/\mu^2 K \tag{3}$$

where K and μ are constants for a particular set of T-networks.

The β of such a multiple-T network is given by the formula,

$$\beta = \frac{n_r \mu^2 + S^2 n_c C^2 R^2}{n_r \mu^2 + S^2 n_c C^2 R^2 + SCR \dfrac{(1+K)}{K} (n_r + \mu^2 n_c)} \tag{4}$$

where n_r and n_c are the number of resistance and capacitance arms respectively in the multiple-T.

Substituting

$$x = n_r \mu^2$$

$$y = n_c C^2 R^2$$

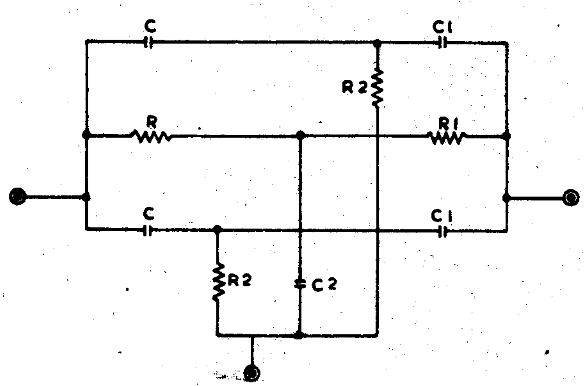

Fig 1—RC multiple-T network.

and $z = CR \dfrac{(1 + K)}{K} (n_r + \mu^2 n_c)$

we get

$$\beta = \frac{x + S^2 y}{x + S^2 y + Sz} \tag{5}$$

$$V_0(S) = \frac{1}{S} \; V_i \; (S)$$

$$\frac{x + S^2 y}{S\,(x + S^2 y + Sz)}$$

Solving the above equation, we get the output voltage V_0

$$V_0 = 1 - \frac{ze^{-zt}}{(z^2 - 4x)^{\frac{1}{2}}} \left[e^{(z^2 - 4x)^{\frac{1}{2}} \, t/2} - e^{(z^2 - 4x)^{\frac{1}{2}} \, t/2} \right]$$

This equation is true for the three conditions given by

$z^2 > 4x$, which corresponds to over damping

$z^2 = 4x$ which corresponds to critical damping and

$z^2 < 4x$ which corresponds to under damping

RESULT AND DISCUSSION

Case I—When $z^2 > 4x$

$$C^2 R^2 \left(\frac{1 + K}{K}\right)^2 (n_r + \mu^2 n_c)^2 > 4n_r \, \mu^2$$

$$CR > \frac{2\mu \, K \, (n_r)^{\frac{1}{2}}}{(1+K)\,(n_r + \mu^2 n_c)} \tag{8}$$

Table 1 gives the lower limit for the *CR* values for different multiple-Ts which satisfy the condition of over damping, taking $\mu = K = 1$.

It is seen from the results that as the number of T-sections $(n_r + n_c)$ increases, the lowest value of the time constant required to satisfy the condition necessary for over damping decreases. It is also found that as n_r decreases and n_c correspondingly increases, the lowest value of *CR* also decreases and hence the condensers in the multiple-Ts discharge quickly and reach almost zero voltage condition earlier. Fig. 2 shows the discharge curves for various multiple-Ts relatively. All the curves are arranged in a decreasing order of their time constants. OA is the step voltage input for which the output AC is obtained. For the same input, the outputs for different multiple-Ts are as shown in Fig. 2.

From the graphs it can be easily seen that as the number of resistance arms increased from one to four, the lowest value of the time constant decreases. But when the value of n_c=2, the value of the lowest time constant increases, for an increase of n_r and then decreases for a further increase of n_r. When n_c=3, the lowest time constant increases for an increase of n_r and then decreases for a further increase of n_r.

TABLE 1

Lower limit for *CR* values different multiyle Ts

T-net works		n_r	n_c	$CR > (n_r)^{\frac{1}{2}}/(n_r + n_c)$
Twin-T		1	1	·500
Triple-T	(a)	2	1	·471
	(b)	1	2	·333
Quadruple-T	(a)	3	1	·433
	(b)	2	2	·354
	(c)	1	3	·250
5-Ts	(a)	4	1	·400
	(b)	3	2	·346
	(c)	2	3	·283
	(d)	1	4	·200

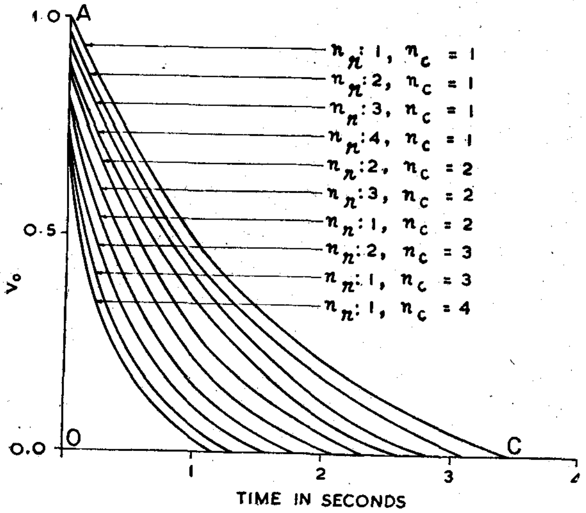

Fig 2—Overdamping in multiple-Ts.

Ca e II—When $z^2 = 4x$ (critical damping)

$$V_0 = 1 - \frac{z}{0} \tag{9}$$

The result breaks down because the coefficients become infinite.

let $\quad (z^2 - 4x)^{\frac{1}{2}} = h$ (a very small quantity) $\quad$ (10)

then

$$V_0 = 1 - \frac{z}{h}\, e^{-zt} \left[e^{(h/2)t} - e^{(h/2)t} + \ldots . \right] \tag{11}$$

$$= 1 - \frac{ze^{-zt}}{h} \left[ht + \frac{h^3 t^3}{4\,3} + \ldots .. \right] \tag{12}$$

Since h is very small, terms involving h^3 and higher orders can be neglected.

Hence
$$V_0 = 1 - zte^{-zt} \tag{13}$$

$$z = CR\, \frac{(1+K)}{K} (n_r + \mu^2 n_c) \tag{14}$$

When $\quad \mu = K = 1,\ z = 2CR\,(n_r + n_c)$

For a twin-T, $n_r = n_c = 1$, hence $z = 4CR$. Similarly, for a triple-T, $z{=}6CR$, quadruple-T, $z{=}8CR$ ets For a unit step input, the output voltage V_0 can be plotted for any multiple-T by giving different valueo for t. Fig. 3 gives a series of curves for a twin-T, triple-T, a quadruple-T etc. having a CR value equaן to 0·51. It is found from the curves obtained that the time taken to reach the steady state in the case of the twin-T is the longest and goes on decreasing for increasing number of Ts as long as the T-combination behaves as a tuned circuit. For a given time constant which satisfies the above relationship, the shape of the curve for a given number of Ts remains unaltered whatever may be the nature of the Ts as long as the resultant T-combination acts as a tuned circuit. In other words, it is independent of the number of capacitance arms and the resistance arms as long as the total is constant and the time constant chosen is greater than that required for the particular combination given in Table 1. The output voltage V_0 reaches the same minimum value corresponding to a value of $zt = 1$, whatever may be the number of Ts as long as the combination behaves as a tuned circuit. As the number of Ts is increased, the minimum value of V_0 and the steady state value are reached earlier.

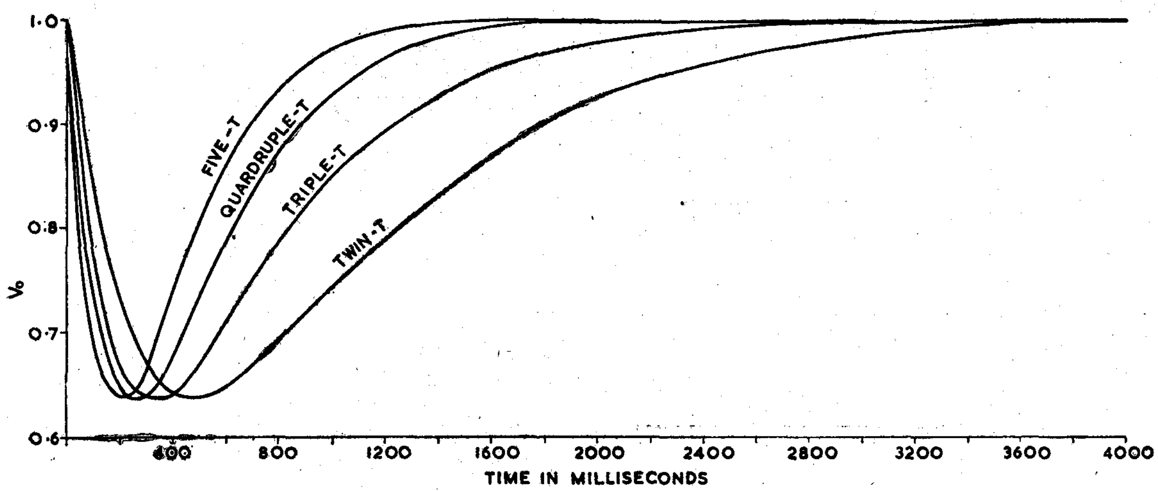

Fig 3—Critical damping in multiple-Ts.

The frequency of the multiple-T is given by

$$f_m = f_d \left(\frac{n_r}{n_c} \right)^{\frac{1}{2}} \tag{15}$$

where

$$f_d = \frac{1}{2\pi\, CR} \text{ for a twin-T}$$

But the minimum value of CR is given by (8)

$$CR = \frac{2\, \mu\, K\, (n_r)^{\frac{1}{2}}}{(K+1)\,(n_r + \mu^2 n_c)}$$

This value of CR corresponds to the maximum frequency f_m. Therefore, the maximum resonant frequency for the multiple-T to satisfy the critical damping condition is given by

$$f_m = \frac{(K+1)\,(n_r + \mu^2 n_c)}{4\, \pi\, \mu\, K \sqrt{n_c}} \tag{16}$$

When

$$\mu = K = 1,\; f_m = \frac{(n_r + n_c)}{2\pi \sqrt{n_c}} \tag{17}$$

Case III—When $z^2 < 4x$ (under damping)

The output

$$V_0 = 1 - \frac{2z\, e^{-zt}}{(4\,x - z^2)^{\frac{1}{2}}} \sin \left[\frac{(4x - z^2)\;\; t}{2} \right] \tag{18}$$

when

$$t = 0,\; V_0 = 1$$

As 't' is increased, it is found that a damped sine wave with the mean level above the input voltage is obtained as shown in Fig. 4.

The period of the sine wave is given by

$$4\pi/(4x - z^2)^{\frac{1}{2}}$$

The frequency of the damped oscillation is given by

$$W_{DA} = \frac{(4\,x - z^2)^{\frac{1}{2}}}{2} \tag{19}$$

$$= \frac{\left\{ 4n_r\, \mu^2 - C^2 R^2 \dfrac{(1+K)^2}{K} (n_r + \mu^2\; n_c)^2 \right\}^{\frac{1}{2}}}{2} \tag{20}$$

The resonant angular frequency is given by

$$W_0 = \frac{\mu}{CR} \left(\frac{n_r}{n_c} \right)^{\frac{1}{2}} \tag{21}$$

and

$$Q = \frac{\mu K\, (n_r\, n_c)^{\frac{1}{2}}}{(K+1)\,(n_r + \mu^2 n_c)} \tag{22}$$

$$\therefore\; W_{DA} = \frac{\mu}{2} \left\{ n_r \left(4 - \frac{\mu^2\, n_r}{W_0^2\; Q^2} \right) \right\}^{\frac{1}{2}} \tag{23}$$

When $\mu = K = 1$,

then from (20) we have

$$W_{DA} = \{ n_r - C^2 R^2 (n_r + n_c)^2 \}^{\frac{1}{2}} \tag{24}$$

For W_{DA} to be positive, n_r is greater than $C^2R^2\,(n_r + n_c)^2$

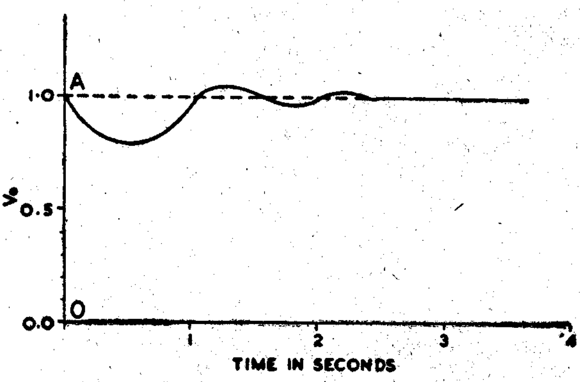

Fig 4—Under damping in multiple-Ts.

or
$$CR < \frac{(n_r)^{\frac{1}{2}}}{(n_r+n_c)}$$

W_{DA} increases as the value of CR decreases which condition is also satisfied for resonance. For a given total number of Ts, (n_r+n_c), the number of cycles in the damped oscillation depends on the value of n_r. Greater the value of n_r, greater is the frequency. Thus for an increase of n_r, there is an increase in the number of cycles per second.

Time taken to reach the steady state depends on the amplitude term $2\,ze^{-z}/(4\,x-z^2)^{\frac{1}{2}}$. For an increase in the value of $(4\,x-z^2)^{\frac{1}{2}}$ the frequency increases whereas the amplitude decreases. Considering equation (23) for a twin-T, as W_0 increases, $(1/W_0^2\,Q^2)$ decreases and hence the factor $(4-1/W_0^2\;Q^2)^{\frac{1}{2}}$ in the equation for damped oscillation for a twin-T increases and W_{DA} increases. A twin-T has a value for $Q=\frac{1}{4}$ and hence $W_{DA}=(1-4/W_0^2)^{\frac{1}{2}}$. For large values of W_0, $W_{DA}=1$. Therefore, the damped oscillation has a maximum angular frequency equal to one in the case of a twin-T.

Table 2 gives the Q values, limits of W_0 and the corresponding limits of W_{DA}. It is seen that the angular frequency of the damped oscillations decreases with the decrease of the value of n_r and a corresponding increase in the value of n_c in the symmetrical RC parallel multiple-T. There is a lower limit for the

TABLE 2

LIMIT OF W_{DA} FOR DIFFERENT MULTIPLE-Ts

T-network		n_r	n_c	Q	Limits of w_0	Corresponding limits of W_{DA}
Twin-T		1	1	·250	∞ to 2·000	1·000 to 0
Triple-T	(a)	2	1	·236	∞ to 3·200	1·414 to 0
	(b)	1	2	·236	∞ to 2·120	1·000 to 0
Quadruple-T	(a)	3	1	·217	∞ to 4·000	1·732 to 0
	(b)	2	2	·250	∞ to 2·828	1·414 to 0
	(c)	1	3	·217	∞ to 2·301	1·000 to 0
5-Ts	(a)	4	1	·200	∞ to 5·000	2·000 to 0
	(b)	3	2	·245	∞ to 3·535	1·732 to 0
	(c)	2	3	·245	∞ to 2·885	1·414 to 0
	(d)	1	4	·200	∞ to 2·500	1·000 to 0

resonant angular frequency $W_0=(n_r)^{\frac{1}{2}}/2Q$ for each multiple-T at which the damped angular frequency W_{DA} becomes zero. When W_0 becomes very very high so that the factor $n_r/W_0^2\,Q^2$ becomes negligible compared with four in equation (23), W_{DA} becomes equal to n_r. The ratio of this value of W_{DA} to twice the value of the lower limit of W_0 gives the value of the Q of the multiple-T given. This gives a method of producing very very low frequency damped oscillations using a twin-T or a multiple-T. Since the damped frequency of the oscillations is very very low, they can be displayed only on a equally low frequency time-base of an oscilloscope.

Comparison Between a Series Tuned LCR Circuit and a Symmetrical RC Parallel Multiple-T

In general, a symmetrical RC parallel multiple-T behaves as a series tuned circuit. In both cases, the amplitude and the phase minima occur at the same resonant frequency. The output waveform from a symmetrical RC multiple-T for a step input does not correspond to the output waveform from across the inductance of a series tuned circuit. So far as the initial step is concerned both the outputs are the same, whereas the steady state value in the case of the output across the inductance has a zero value and in the case of the multiple-T, it corresponds to the step voltage. In the case of a series tuned LCR circuit, the damped oscillation frequency is given in terms of W_0, by

$$W_{DA}=W_0\,(1-1/Q^2)^{\frac{1}{2}}$$

whereas for a symmetrical multiple-T, it is given by

$$W_{DA}=\frac{u}{2}\;\left\{n_r\;(4-\mu^2\,n_r/W_0^2Q^2)\right\}^{\frac{1}{2}}$$

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