DUSTY VISCOUS FLOW THROUGH THE ANNULUS BETWEEN TWO COAXIAL CIRCULAR CYLINDERS

E. RUKMANGADACHARI

Department of Mathematics
P. R. Govt. College Kakinada-533001

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Unsteady laminar flow of a dusty viscous incompressible fluid through the annular tube formed by two coaxial circular cylinders is discussed under two situations : (i) when the pressure gradient varies harmonically with time; and (ii) when there is an exponential pressure gradient. The flow through a circular pipe is obtained as a particular case, making the radius of the inner cylinder tend to zero.

The study of the motion of dusty viscous fluid is not merely interesting in itself but has application in a wide variety of situations. Such situations arise, for instance, in the movement of dust-laden air, in fluidization, in the use of dust in gas cooling system and in sedimentation in tidal waves.

Several studies have already been made by various authors[1] to understand the effect of the presence of dust, using Saffman's model[2].

In this paper, we consider the motion of unsteady, incompressible dusty viscous fluid through the annulus formed by two coaxial circular cylinders in two cases : (i) when the pressure gradient varies harmonically with time and (ii) when the flow is subjected to an exponential pressure gradient; and the results for the case of flow through a circular cylinder are obtained by making the radius of the inner cylinder tend to zero.

GOVERNING EQUATIONS

The equations governing the motion of an unsteady laminar flow of a dusty, viscous, incompressible fluid as given by Saffman[2] are

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}.\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{u} + \frac{KN}{\rho}(\bar{v} - \bar{u}) \tag{1}$$

$$\operatorname{div}\bar{u} = 0 \tag{2}$$

$$m\left[\frac{\partial \bar{v}}{\partial t} + (\bar{v}.\nabla)\bar{v}\right] = K(\bar{u} - \bar{v}) \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div}(N\bar{v}) = 0 \tag{4}$$

where $\bar{u}$ and $\bar{v}$ denote the local velocity vectors of fluid and dust particles respectively, ρ is the fluid density, p the fluid pressure, ν the kinematic viscosity, N the number density of dust particles, K the Stokes resistance coefficient (for spherical particles of radius ϵ it is $6\pi\mu\epsilon$), μ the fluid viscosity and m the mass of a dust particle.

Consistent with the geometry of the problem, we take the cylindrical polar coordinate system (z, r, θ) so that z-axis coincides with the common axis of the cylinders. The flow is directed along this axis, and is symmetrical w.r.t. it. Consequently the velocity vectors $\bar{u}$ and $\bar{v}$ have only z-components. If u, v denote the component of velocity of the fluid and that of the dust particles respectively in z-direction, we have from (2) and (4)

$$\frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial v}{\partial z} = 0 \tag{5}$$

where by symmetry, u and v are independent of θ and the number density N of the dust particles is assumed to be constant N_0 throughout the motion. Thus, we see that u and v are functions of the radial distance r only.

The equations of motion then reduce to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{1}{\tau}(v-u) \tag{6}$$

and

$$\tau \frac{\partial v}{\partial t} = u - v \tag{7}$$

where

$$\tau = \frac{m}{K} \text{ and } f = \frac{mN_0}{\rho} \tag{8}$$

are the relaxation time and mass concentration of dust particles, respectively.

SOLUTION OF PROBLEM

Case (*i*) : Firstly, we consider that the pressure gradient varies harmonically with time. Assuming axial symmetry, we take

$$u = \phi(r)\, e^{-i\omega t} \tag{9}$$

$$v = \psi(r)\, e^{-i\omega t} \tag{10}$$

$$\frac{\partial p}{\partial z} = p_0 e^{-i\omega t} \tag{11}$$

where ϕ and ψ involve only r and p_0 is a real constant.

Substituting (9) to (11) in (6) and (7), we obtain

$$\frac{d^2\phi(r)}{dr^2} + \frac{1}{r}\frac{d\phi(r)}{dr} + \lambda^2\phi(r) = Q \tag{12}$$

and

$$\psi(r) = \left(\frac{1}{1-i\omega\tau}\right)\phi(r) \tag{13}$$

where

$$\lambda^2 = \frac{i\omega}{\nu}\left(\frac{1+f-i\omega\tau}{1-i\omega\tau}\right) \tag{14}$$

and

$$Q = p_0/\mu \tag{15}$$

The boundary conditions for the flow through the annular pipe formed of two coaxial circular cylinders with radii, $r=b$ and $r=a$ $(0 < b < a < \infty)$ are

$$\phi = 0, \qquad \text{when } r = a, b \tag{16}$$

Writing $\phi = x(x) + Q/\lambda^2$ equation (12) can be transformed into

$$\frac{{}^2X}{dx} + \frac{1}{x}\frac{dX}{dx} + X = 0 \tag{17}$$

where $x = \lambda r$.

Solving (17) we have

$$\phi = A\,J_0(\lambda r) + B Y_0(\lambda r) + (Q/\lambda^2) \tag{18}$$

where J_0 and Y_0 are Bessel functions of the first and second kind, both of order zero and A and B are constants. The boundary conditions (16) yield

$$A = \frac{Q}{\lambda^2 D}\left[Y_0(\lambda a) - Y_0(\lambda b)\right] \tag{19}$$

$$B = -\frac{Q}{\lambda^2 D}\left[J_0(\lambda a) - J_0(\lambda b)\right] \tag{20}$$

where

$$D = J_0(\lambda a)\,Y_0(\lambda b) - J_0(\lambda b)\,Y_0(\lambda a) \tag{21}$$

We have, in fact[2]

$$J_0(\lambda r) = \sum_{k=0}^{\infty} (-1)^k (\lambda r)^{2k} / 2^{2k} (k!)^2$$

Consequently, we have the solutions

$$\phi = \frac{Q}{\lambda^2 D} \triangle , \psi = \frac{Q}{\lambda^2 D} \frac{\triangle}{1 - i \omega \tau} \tag{22}$$

where

$$\triangle = \Big[Y_0(\lambda a) - Y_0(\lambda b) \Big] J_0(\lambda r) - \Big[J_0(\lambda a) - J_0(\lambda b) \Big] Y_0(\lambda r) + $$
$$+ \Big[J_0(\lambda a) Y_0(\lambda b) - Y_0(\lambda a) J_0(\lambda b) \Big] \tag{23}$$

and therefore the velocity components u and v of the fluid and dust particles respectively are

$$u = \frac{Q}{\lambda^2} \cdot \frac{\triangle}{D} \cdot e^{-i\omega t} \tag{24}$$

$$v = \frac{Q}{\lambda^2} \cdot \frac{\triangle}{D} \left(\frac{1}{1 - i \omega \tau} \right) e^{-i \omega t} \tag{25}$$

where D and $\triangle$ are given respectively by (21) & (23).

Case (*ii*) : Now, we consider that pressure gradient varies exponentially with respect to time and assuming axial symmetry we take.

$$u = \phi(r) e^{-\sigma t} , \tag{26}$$

$$v = \psi(r) e^{-\sigma t} , \tag{27}$$

$$\frac{\partial p}{\partial z} = p_0 e^{-\sigma t} \tag{28}$$

Working as before, we obtain

$$u = \frac{Q}{\lambda_1^2} \cdot \frac{\triangle_1}{D_1} \cdot e^{-\sigma t} ; v = \frac{Q}{\lambda_1^2} \cdot \frac{\triangle_1}{D_1} \cdot \frac{1}{1 - \sigma \tau} e^{-\sigma t} \tag{29}$$

where

$$\lambda_1^2 = \frac{\sigma}{\nu} \left(\frac{1 + f - \sigma \tau}{1 - \sigma \tau} \right)$$

and D_1 and $\triangle_1$ are obtained replacing λ by λ_1 in (21) and (23) respectively.

The case when the pressure gradient decreases exponentially with time, can be considered similarly by taking σ instead of $-\sigma$; σ being taken positive in both the cases.

PARTICULAR CASES

Writing ϕ as

$$\phi = \frac{Q}{\lambda^2} \left\{ 1 + \frac{J_0(\lambda r) \left(\frac{Y_0(\lambda a)}{Y_0(\lambda b)} - 1 \right) - \frac{Y_0(\lambda r)}{Y(\lambda b)} \Big(J_0(\lambda a) - J_0(\lambda b) \Big)}{\Big[J_0(\lambda a) - J_0(\lambda b) \frac{Y_0(\lambda a)}{Y_0(\lambda b)} \Big]} \right\}$$

$$(0 < b < a < \infty) \tag{30}$$

and noticing that[3]

as $b \rightarrow 0, \quad Y_0(\lambda b) \rightarrow -\infty$

we get

$$\phi = \frac{Q}{\lambda^2}\left[1 - \frac{J_0(\lambda r)}{J_0(\lambda a)}\right] \tag{31}$$

which agrees with the corresponding result got by Kishore & Pandey[4] for the flow of dusty viscous liquid through a circular pipe and therefore the velocity fields tally in this limiting case, $r=b$ excepted, being a point of singularity.

It is obvious that similarly one can obtain results in Case (*ii*) too.

FLUX AND DRAG

We now consider the flux and Skin friction drag due to the motion of dusty viscous fluid through a circular cylinder, using the expression for the velocity of the fluid given by (31).

The flux per unit length is given by

$$Q = \int_0^a \int_0^{2\pi} u r d\theta\, dr = 2\pi \int_0^a u\, r\, dr \tag{32}$$

where u, the velocity of the fluid in the axial direction, is independent of θ ; and the formula for the drag acting on the curved surface, per unit length of the cylinder is

$$D = \int_c \tau_{rz}\, ds$$

where τ_{nz} is the z-component of the thrust on a surface whose outward-drawn normal is n and ds is the elemental arc length. In our case,

$$D = \mu \int_0^{2\pi} \left(r\frac{\partial u}{\partial r}\right)_{r=a} d\theta = 2\pi\mu a \left(\frac{\partial u}{\partial r}\right)_{r=a} \tag{33}$$

If we retain terms up to $0(|\lambda|^2)$, we get

$$Q = \frac{\pi a^4}{8\mu} p_0 e^{-i\omega t}$$

$$D = \pi a^2 p_0 e^{-i\omega t}$$

which expressions are those of the clean fluid[5]. Thus it is observed that the influence of the presence of dust is not felt up to $0(|\lambda|^2)$. Hence, retaining terms up to $0(|\lambda|^4)$ in the Bessel function expansions we find that Q and D may be put as follows when we write the real parts only.

$$\begin{aligned} Q &= Q_c + Q_d \\ &= \left[(k\omega^{-1}\cos\omega t/8\mu) + (k a^2 \sin\omega t/96\mu\nu)\right]_c + \\ &\quad + \left[-ka^2 f\beta^{-1}\cos(\omega t+\alpha)/96\mu\nu\right]_d \end{aligned} \tag{34}$$

$$\begin{aligned} D &= D_c + D_d \\ &= \left[k\omega^{-1}\cos\omega t - (k/4\nu)\sin\omega t)\right]_c + \\ &\quad + \left[kf\beta^{-1}\cos(\omega t+\alpha)/4\nu\right]_d \end{aligned} \tag{35}$$

where

$$k = \pi a^4 \omega p_0,\ \alpha = \cot^{-1}\omega\tau,\ \beta = (1+\omega^2\tau^2)^{\frac{1}{2}}$$

$[\ ]_c$ is the clean viscous flow part; and $[\ ]_d$ is the dusty viscous flow part of the flux or drag, as the case may be.

We observe that the presence of the dusty particles in the fluid is to decrease the flux and to increase the drag on the walls of the cylinder by amounts

$$Q_d = k\, a^2 f \beta^{-1} \cos(\omega t + \alpha)/96\, \mu\nu$$

and
$$D_d = (k / 4\nu) f \beta^{-1} \cos(\omega t + \alpha) \tag{36}$$

respectively, as can be expected from physical considerations.

Now, for small values of τ, we have

$$\beta^{-1} = (1 + \omega^2 \tau^2)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}\, \omega^2 \tau^2$$

which shows that the additional drag due to the presence of dust decreases as τ increases up to a critical level given by $\tau_{crit} = \sqrt{2}/\omega$. This agrees with Saffman's observation that when τ is small, which corresponds to the case of the dust being fine, the effective kinematic viscosity is reduced. Again, for large τ, we have

$$f\beta^{-1} = f(\omega\tau)^{-1} \left(1 + \frac{1}{\omega^2 \tau^2}\right)^{-\frac{1}{2}} \simeq f(\omega\tau)^{-1} \left(1 - \frac{1}{2\omega^2 \tau^2}\right)$$
$$= s\,\omega^{-1} \left[1 + 0\{(\omega\tau)^{-2}\} \right]$$

where $s = f\tau^{-1} = \frac{KN_0}{\rho}$; the additional drag D_d varies with $s\,\omega^{-1}$ in the case of coarse dust.

For the annulus between two coaxial cylinders the flux is

$$Q_A = \frac{a^2 e^{-i\omega t}}{24} \Big[(24\pi Q/\lambda^2) + 3\pi A(8 - \lambda^2 a^2) +$$
$$+ 2B\left\{ \lambda a^2 (4 - 3a) + (3 \log \tfrac{1}{2}\lambda a + \gamma)(8 - \lambda^2 a^2) \right\} \Big]$$

where γ is the Euler's constant. The drag per unit length on the curved surface of the outer cylinder $r=a$ is

$$D_A = 4\mu\pi\lambda a \left[A J'_0(\lambda a) + B Y'_0(\lambda a) \right].$$

DEDUCTIONS

1. Removal of the dust particles from the expressions by making $f = 0$ results in all the flow quantities of nondusty viscous flows being regained.

2. A general analytical discussion of the velocity expressions got for the annular tube being difficult, we confine the discussion to two situations of the particular case discussed in the section of particular cases in this paper viz. when the frequency is very small and when it is very large.

If mass concentration f is large, retaining terms up to $0(f^{-1})$, we have

$$\frac{1}{\lambda^2} = \frac{\nu}{i\omega}\left(\frac{1 - i\omega\tau}{1 + f - i\omega\tau}\right) \simeq -\frac{i\nu}{\omega(1+f)} - \frac{\nu\tau}{1+f} \tag{37}$$

For oscillatory flow through a circular cylinder, we have found the velocity of the fluid at (31) which can be put as

$$u(r, t) = -i \frac{\rho p_0}{\omega(1+f)} e^{-i\omega t} \left[1 - \frac{J_0(\lambda r)}{J_0(\lambda a)} \right] \tag{38}$$

Expanding the Bessel function in a series and retaining only up to the quadratic terms we obtain the following expression which is valid for the case of very small values of the dimensionless group $\sqrt{\omega/\nu}\, a$ which corresponds to very slow oscillations.

$$u(r, t) = -i \frac{\rho p_0}{\omega(1+f)} e^{-i\omega t} \left[1 - \frac{1 - \frac{i\omega}{4\nu} r^2}{1 - \frac{i\omega}{4\nu} a^2} \right] \tag{39}$$

Here, we have considered τ to be large which means that the dust particles are coarse so that

$$\lambda^2 = \frac{i\,\omega}{\nu}\left(\frac{1+f-i\,\omega\,\tau}{1-i\,\omega\,\tau}\right) \simeq \frac{i\,\omega}{\nu} \tag{40}$$

Now equation (34) reduces finally to

$$u(r,t) = -\frac{\rho\,p_0}{4\,\nu}\,\frac{1}{(1+f)}\,(a^2-r^2)\cos\omega t \tag{41}$$

where we have taken only the real part on the RHS.

Again, using the asymptotic formula for the Bessel function expansion[6]

$$J_0(Z) \sim \sqrt{2/\pi Z}\; e^{iZ}\,i^{-\frac{1}{2}}$$

we obtain the velocity expression for very large values of $\sqrt{\omega/\nu}\,a$, as

$$u(r,t) = i\,\frac{\rho p_0}{\omega(1+f)}\,e^{-i\omega t}\left[1-\sqrt{\frac{a}{r}}\exp\left\{-(1-i)\sqrt{\frac{\omega}{2\nu}}\,(a-r)\right\}\right]$$

$$= -\frac{\rho p_0}{\omega(1+f)}\left[\sin\omega t-\sqrt{\frac{a}{r}}\left\{\exp-\sqrt{\frac{\omega}{2\nu}}\,(a-r)\right\}\sin\left\{\omega t-\sqrt{\frac{\omega}{2\nu}}\,(a-r)\right\}\right] \tag{42}$$

where we have taken the real part only. Provided that we put $f=0$ in the RHS of the (41) and (42) they agree with the corresponding velocity expressions (4) for the oscillatory, non-dusty viscous flow through a circular pipe.

It may be observed that in this case namely, when τ is very large, the velocity of the dust particles vanishes both in slow and large oscillations as it can be expected since the fluid will not be able to move particles of large masses.

In case τ is small

$$\lambda^2 = \frac{i\,\omega}{\nu}\left(\frac{1+f-i\,\omega\,\tau}{1-i\,\omega\,\tau}\right) \simeq \frac{i\,\omega}{\nu}\,(1+f) \tag{43}$$

we have for small oscillations $\left[1-\frac{J_0(\lambda r)}{J_0(\lambda a)}\right] \simeq -\frac{1}{4}\,\frac{i\,\omega}{\nu}\,(1+f)\,(a^2-r^2).$

So, the velocity of the fluid in this case is

$$u = -\frac{\rho p_0}{4\,\nu}\,(a^2-r^2)\cos\omega t \tag{44}$$

where we have taken the real part only on the RHS.

This shows that the velocity distribution for small τ is basically same as for clean fluid which is in agreement with the observation made by Saffman.

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