STRESSES IN AN ISOTROPIC ELASTIC PLATE IN THE FORM OF PASCAL'S LIMACON UNDER CONCENTRATED FORCES

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The problem of a thin elastic plate in the form of Pascal's limacon under concentrated forces at the extremeties of its axis has been solved by using complex variable technique. The solution has been obtained in a closed form. Stress components have been found out. In particular, the solution of an elastic circular plate and that of a plate in the form of a cardioid have been discussed. The variation of stress—intensity factor has been studied.

Complex variable methods to solve two-dimensional boundary value problems in elasticity have been developed by Muskhelishivilli[1]. Later on, these methods were exploited by various authors [2,3,4&5] to solve variety of problems. Milne-Thomson[4] has, however, used the method of analytic continuation across the boundary of the plate. Using this method he has solved the problem of an epitrochoidal oval under two standard concentrated forces at the ends of its 'major' axis.

In the present paper, the same method has been employed to solve the problem of a thin isotropic elastic plate in the form of Pascal's limacon under two standard concentrated forces at the extremeties of its axis. The solution has been obtained in a closed form. Solutions for a circular plate and that for a plate in the form of a cardioid can be obtained as particular cases.

FUNDAMENTAL FORMULAE

The boundary of a thin homogeneous isotropic plate in the form of Pascal's limacon is assumed under the action of two standard concentrated loads at the ends of its axis. We denote the boundary of the plate by γ and the region inside and outside it by L_1 and R_1 respectively.

The transformation function

$$z = m\,(\xi) = c(\xi + K\xi^2) \tag{1}$$

where

$$c > 0,\ 0 \leqslant K \leqslant \tfrac{1}{2},\ z = \rho e^{i\phi} \quad , \ \xi = re^{i\theta}$$

maps the boundary γ of the limacon in the z-plane on to a unit circle Γ in the ξ-plane. The region inside and outside ν are mapped on to those inside and outside Γ which are denoted by L and R respectively.

Determination of two potential function $W(\xi)$ and $w(\xi)$ gives the complete solution of the problem. Stress components $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{\theta\theta}$ are given in term of these two potential functions as follows[4].

$$2\,(\widehat{rr} + i\,\widehat{r\theta}\,) = W(\xi) + \overline{W}\,(\overline{\xi}) - \frac{\overline{\xi}\, m(\xi)}{\xi\, m'(\xi)} N^1(\xi)\ \overline{W'}\ \frac{\overline{\xi}\ \overline{m'}\ (\overline{\xi})}{\xi\ m'\ (\xi)}\ \overline{w}\ (\overline{\xi}) \tag{2}$$

$$(2\ \widehat{\theta\theta}\ - i\ \widehat{\theta r} = W(\xi) + \overline{W}\ (\overline{\xi}) + \frac{\overline{\xi}\ m(\xi)}{\xi\ m'(\xi)}\ \overline{W}'\ (\overline{\xi}) + \frac{\overline{\xi}'\ \overline{m}'\ (\overline{\xi})}{\xi\ m'\ (\xi)}\ \overline{w}\ (\overline{\xi})\ . \tag{3}$$

Following Milne-Thomson, we apply the continuation theorem for the circle to equation (2) to give

$$m'(\xi)\ W(\zeta) \ = \ \frac{1}{2\pi i}\int \frac{2\left[\ -p(\sigma) + i\ s(\sigma)\ \right]\ m'\,(\sigma)}{\sigma - \xi}\ d\sigma + \psi\ (\xi) \tag{4}$$

where $\sigma = e^{i\theta}$, $p(\sigma)$ is the pressure, $s(\sigma)$ is the shear and $\psi(\xi)$ is continuous across Γ and its form is obtained by considering the singularities of $m'(\xi)\ \overline{W}(\overline{\xi})$ in the region R. The analytic continuation in R of (2) is obtained by writing zero for $\widehat{rr} + i\,\widehat{r\theta}$ and $\frac{1}{\xi}$ for ξ as follows.

$$m'(\xi)\ W(\xi) = -\,m'(\xi)\overline{W}\left(\frac{1}{\xi}\right) + \frac{1}{\xi^2}\,m(\xi)\ \overline{W}'\left(\frac{1}{\xi}\right) + \frac{1}{\xi^2}\,\overline{m}'\left(\frac{1}{\xi}\right)\overline{w}\left(\frac{1}{\xi}\right) \qquad (\xi \text{ in } R) \tag{5}$$

Again, from (5), by writing $\overline{\xi}$ for $\frac{1}{\xi}$ and $\frac{1}{\xi}$ for ξ and taking the complex conjugate, we get

$$m'(\xi)\,w(\xi) = \frac{1}{\xi^2}\ \overline{m}'\left(\frac{1}{\xi}\right)\left[W(\xi) + \overline{W}\left(\frac{1}{\xi}\right)\right] - \overline{m}'\left(\frac{1}{\xi}\right)W'(\xi) \qquad (\xi \text{ in } L) \tag{6}$$

The function $w(\xi)$ must be holomorphic in L. The equation (6) will be used to express the holomorphy condition which will enable us to find the unknown constants of $\psi(\xi)$ in (4). However, we can dispense with the function $w(\xi)$ expressing stresses in terms of $W(\xi)$ and its continuation in R. Taking the complex conjugate of (6) and then eliminating $\overline{w}\,(\overline{\xi})$ from (2) and (3), we get

$$2\xi\ m'(\xi)\left[\widehat{rr} + i\,\widehat{r\theta}\right] = \xi\,m'(\xi)\,W(\xi) - \frac{1}{\xi}\,m'\left(\frac{1}{\xi}\right)W\left(\frac{1}{\xi}\right) + $$

$$+\left[\xi\,m'(\xi) + \frac{1}{\overline{\xi}}\,\overline{m}'\left(\frac{1}{\overline{\xi}}\right)\right]\overline{W}(\overline{\xi}) + \left[m(\xi) - \overline{m}\left(\frac{1}{\overline{\xi}}\right)\right]\overline{\xi}\ \overline{W}'(\overline{\xi}) \tag{7}$$

$$2\xi\ m'(\xi)\left[\widehat{\theta\theta} + i\,\widehat{\theta r}\right] = \xi m'(\xi)\,W(\xi) + \frac{1}{\xi}\,m'\left(\frac{1}{\xi}\right)W\left(\frac{1}{\xi}\right) + $$

$$+\left[\xi\,m'(\xi) - \frac{1}{\overline{\xi}}\,\overline{m}'\left(\frac{1}{\overline{\xi}}\right)\right]\overline{W}(\overline{\xi}) + \left[m(\xi) - \overline{m}\left(\frac{1}{\overline{\xi}}\right)\right]\overline{\xi}\ \overline{W}'(\overline{\xi}) \tag{8}$$

The displacement components u and v, in a similar manner, are given by

$$4\mu\,\frac{\partial}{\partial\theta}\left(u + iv\right) = i\,\chi\,\xi m'(\xi)\,W(\xi) + \left[\frac{1}{\overline{\xi}}\,\overline{m}'\left(\frac{1}{\overline{\xi}}\right) - \xi\,m'(\xi)\right]i\ \overline{W}(\overline{\xi}) + $$

$$+\ \frac{1}{\overline{\xi}}\,m'\left(\frac{1}{\overline{\xi}}\right)i\,W\left(\frac{1}{\overline{\xi}}\right) + \left[m(\xi) - \overline{m}\left(\frac{1}{\overline{\xi}}\right)\right]i\,\xi\ \overline{W}'(\overline{\xi}) \tag{9}$$

where $x = \dfrac{3-\nu}{1+\nu}$, μ and ν are the modulus of rigidity and Poisson's ratio of the material of the plate.

SOLUTION OF THE PROBLEM

The transformation function (1) maps the limacon γ in Z-plane on to a unit circle Γ in ξ-plane. The anti-clockwise sense of description of γ is taken to be positive. As γ is described in Z-plane in the positive sense, Γ-is also described in the same sense in ξ-plane as shown in Fig. 1.

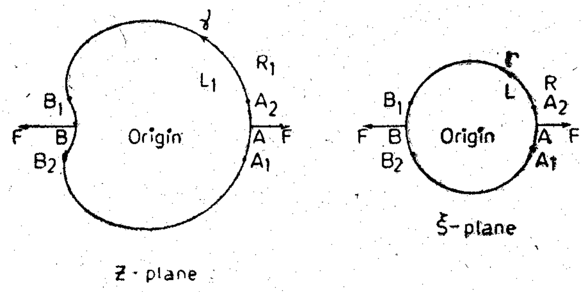

Fig. 1—Transformation of Pascal's Limacon on to a unit circle.

From (1), the parametric equation of the boundary γ of the limacon are given as :

$$x = c\,(\cos\theta + K\cos 2\theta) \tag{10}$$

$$y = c\,(\sin\theta + K\sin 2\theta) \tag{11}$$

where the parameter θ is the vectorial angle of any point on Γ. Points A and B on the contour γ correspond to $\theta=0$ and $\theta=\pi$. Let the distance $c(1+k)$ and $c\,(1-k)$ of A and B from origin be denoted by a and b respectively. The whole boundary γ of the limacon is unloaded except the two points A and B where standard concentrated forces F and $-F$ act to keep the plate in equilibrium. In infinitely small neighbourhood of these points, the stresses are unbounded. Physically this difficulty is resolved by plastic yielding. This being in a very small region near these points, we think these forces to be applied as distribution of stress over small areas round A and B instead of being concentrated at these points. Let ϵ be infinesimal and we take two points A_1, A_2; B_1, B_2 on the limacon given by

$$\left.\begin{aligned} {}^zA_1 = a(1-\epsilon)\,,\ {}^zA_2 = a(1+\epsilon) \\ {}^zB_1 = b(1-i\epsilon)\,,\ {}^zB_2 = -\,b(1+i\epsilon) \end{aligned}\right\} \tag{12}$$

The standard concentrated forces F will be obtained as the limit when $\epsilon\to 0$ of a unifom stress distribution given as

$$\left.\begin{aligned} -\,p(\sigma) + i\,s\,(\sigma) &= \frac{F}{2a\epsilon} \quad \text{on arc } A_1\ A\ A_2 \\ \text{and} \qquad -\,p(\sigma) + i\,s\,(\sigma) &= \frac{F}{2b\epsilon} \quad \text{on arc } B_1\ B\ B_2 \end{aligned}\right\} \tag{13}$$

The form of the unknown $\psi\,(\xi)$ in (4) is obtained by considering the singularities of $m'(\xi)\,W(\xi)$ in R. Since R, contains the point ∞ and $m(\xi) = c\,(\xi + K\xi^2)$ has a pole of order two at $\xi = \infty$, it follows that $m'(\xi)$ $W\,(\xi)$ will have a pole there of order one at most. Hence, we can write

$$\psi\,(\xi) = A_3 + B_3\xi \tag{14}$$

where A_3 and B_3 are unknown constants. The equation (4) reduces to

$$m'\,(\xi)\,W\,(\xi) = \frac{1}{\pi i}\int_{A_1AA_2} \frac{F}{2a\epsilon}\,\frac{c\,(1+2k\sigma)}{\sigma-\xi}\,d\sigma + \frac{1}{\pi i}\int_{B_1BB_2} \frac{F}{2b\epsilon}\,\frac{c\,(1+2K\sigma)}{\sigma-\xi}\,d\sigma + A_3 + B_3\xi$$

which then reduces to

$$m'\,(\xi)\,W\,(\xi) = \frac{Fc}{2\pi a\epsilon i}\Big[\,2K\,({}^{\sigma}A_2 - {}^{\sigma}A_1) + (1+2K\xi)\,l_n\Big\{({}^{\sigma}A_2-\xi)/({}^{\sigma}A_1-\xi)\Big\}\Big] +$$

$$+\ \frac{Fc}{2\pi b\epsilon i}\Big[\,2K\,({}^{\sigma}B_2 - {}^{\sigma}B_1) + (1+2K\xi)\,l_n\Big\{({}^{\sigma}B_2-\xi)/({}^{\sigma}B_1-\xi)\Big\}\Big] \tag{15}$$

Since on the contour γ, $dz = i\epsilon a$ on going from A to A_2 and $dz = i\epsilon b$ on going from B to B_2, thus to the first order we have

$$d\sigma = \frac{i\epsilon a}{c(1+2K)} = f_1(\epsilon), \text{ say, on going from } \sigma_A = 1 \text{ to } \sigma_{A_2}$$

and

$$d\sigma = \frac{i\epsilon b}{c(1+2K)} = f_2(\epsilon), \text{ say, on going from } \sigma_B = -1 \text{ to } \sigma_{B_2}$$

Therefore

$$\left.\begin{aligned} \sigma_{A_1} &= 1 - f_1(\epsilon),\ \sigma_{A_2} = 1 + f_1(\epsilon) \\ \sigma_{B_1} &= -1 + f_2(\epsilon),\ \sigma_{B_2} = -1 - f_2(\epsilon) \end{aligned}\right\} \tag{16}$$

Equation (15) using the logarithmic series in the form $l_n(1+p\epsilon) = p\epsilon$, reduces to

$$c(1+2K\xi)\,W(\xi) = \frac{2F}{\pi}\frac{1}{1-\xi^2} + A_3 + B_3\xi \tag{17}$$

The unknown constants A_3 and B_3 will be found using the holomorphy condition of the function $w(\xi)$ given by (6). By Laurent's expansion, the function $w(\xi)$ can be expressed as

$$cw(\xi) = \frac{D_3}{\xi^3} + \frac{D_2}{\xi^2} + \frac{D_1}{\xi} + g(\xi) \tag{18}$$

where

$$\left.\begin{aligned} D_3 &= 2KA_3 + B_3 + 4KF/\pi \\ D_2 &= 2(1-3K^2)\,A_3 - KB_3 + 2(1-6K^2)\,F/\pi \end{aligned}\right\} \tag{19}$$

$$D_1 = 4K(3K^2-1)\,A_3 + 2K^2 B_3 + 4K(6K^2-1)\,F/\pi$$

and $g(\xi)$ is holomorphic in L. Since $w(\xi)$ is to be holomorphic in L, we must have

$$D_1 = D_2 = D_3 = 0 \tag{20}$$

Equations (20) form a system of three simultaneous equations in two unknown quantities A_3 and B_3. The equations are consistent since the determinant of the coefficients of A_3 and B_3 vanishes. Solving, we get

$$\left.\begin{aligned} A_3 &= -\frac{F}{\pi}\frac{4K^2-1}{2K^2-1} \\ B_3 &= \frac{F}{\pi}\frac{2K}{2K^2-1} \end{aligned}\right\} \tag{21}$$

Since the boundary is unloaded, equation (17) gives $W(\xi)$ for ξ in L and for ξ in R. For $K = 0$, the solution agrees with the known solution for a circular plate. For $K = 1/2$, the limacon of Pascal becomes a cardioid. In this case though $m'(\xi) = 0$ at $\xi = -1$, no contradiction with the general theory arises since in the case of the cardioid the boundary has a cusp[1].

EXPRESSIONS FOR STRESSES

Components of stresses can be easily found from the equations (2) and (3). However, the expressions for stresses on the boundary $r = 1$ are given as follows.

$$\widehat{rr} + \widehat{\theta\theta} = \frac{F f_2 \operatorname{cosec}^2\theta}{\pi c f_1} + \frac{2A_3}{cf_1}(1 + 2K\cos\theta) + \frac{2B_3}{cf_1}(2K + \cos\theta) \tag{22}$$

$$\widehat{\theta\theta} = -\frac{F\operatorname{cosec}^2\theta}{\pi c f_1}\left\{f_2 + 2\sin^2\theta\,(3 + 4K + 2K\cos\theta) + 6K\sin\theta\sin 2\theta\right\} +$$

$$+ \frac{2A_3}{cf_1}(1 + 2K\cos\theta) + \frac{B_3}{2cf_1}(8 + 3\cos\theta) \tag{23}$$

and $$\widehat{r\theta} = \frac{KF \operatorname{cosec}^2 \theta}{2\pi\, c\, f_1} \Big\{ \sin\theta\,(\sin 2\theta + \cos 2\theta) + + (f_2 + 2K \sin\theta - \sin 2\theta - 2K \sin 3\theta)/f_1 \Big\} \tag{24}$$

where $f_1 = 1 + 4K \cos\theta + 4K^2$ and $f_2 = 1 + 2K \cos\theta - \cos 2\theta - 2K \cos 3\theta$.

Stresses in a circular plate and in a plate of the form of a cardioid can be obtained, as particular cases, on substituting $K=0$ and $K=1/2$ respectively in equations (22) to (24).

STRESS INTENSITY FACTOR

The distributions of stress-intensity factor $S = [\widehat{\theta\theta}]_{r=1}$ for some particular cases are studied (Fig. 2).

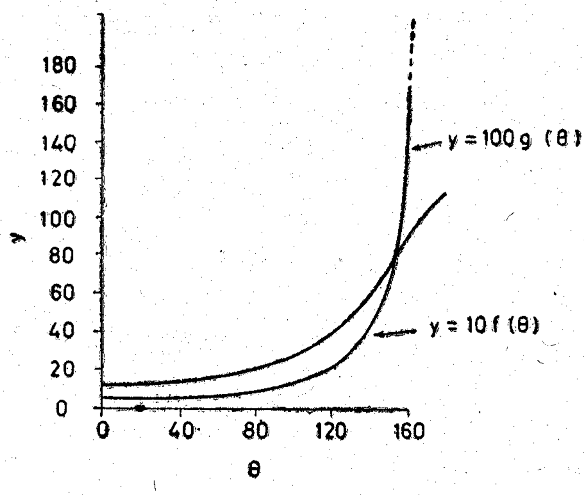

Fig. 2—Variation of stress-intensity factor.

For $K=1/2$, the stress intensity factor S_1 is given as :

$$S_1 = -\frac{F}{c\pi} \left\{ \frac{1 + \cos\theta - \cos 2\theta + \cos 3\theta - 2\cos\theta\cos 2\theta}{1 + \cos\theta - \cos 2\theta - \cos\theta\ \cos 2\theta} \right\} \tag{25}$$

The distribution of S_1 is given in the following table

$$f(\theta) = -c\pi(S_1/F)$$

θ in degrees	20	40	60	80	100	120	140	160
$f(\theta) \times 10$	5.16	5.67	6.67	8.52	12.10	20.00	42.70	165.80

For $K=1/4$, the stress intensity factor S_2 is as follows :

$$S_2 = -\frac{4F}{7c\pi} \left\{ \frac{2 + 7\cos\theta - 2\cos 2\theta + 7\cos 3\theta - 14\cos\theta\ \cos 2\theta}{5 + 4\cos\theta - 5\cos 2\theta - 4\cos\theta\ \cos 2\theta} \right\} \tag{26}$$

$$g(\theta) = -c\pi(S_2/F)$$

θ in degrees	20	40	60	80	100	120	140	160
$g(\theta) \times 100$	13.0	14.1	16.3	20.1	26.5	38.1	59.0	92.1

DISCUSSION

(*i*) Equation (25) shows that S_1 is indeterminate at $\theta=0°$ and $\theta=180°$ where the concentrated forces act. The limiting values of $f(\theta)$ as $\theta \to 0°$ and $\theta \to 180°$ are 0·5 and infinity. At $\theta=180°$, the cardioid has a cusp and the physical impossibility of infinite stress there is resolved by plastic yielding of the material in the small neighbourhood of this point.

(*ii*) Equation (26) shows that S_2 is indeterminate at $\theta=0°$ and $\theta = 180°$. The limiting value of $g(\theta)$ as $\theta \to 0°$ and $\theta \to 180°$ are 0·127 and 1.14 respectively.

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