ON THE TORSION OF ELASTIC HALF-SPACE WITH PENNY-SHAPED CRACK

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(*Received* 28 *August* 1972; *revised* 9 *April* 73)

The investigation deals with the effect of an embedded flaw—a penny-shaped crack in an elastic half-space subjected to torsional deformation. The problem is reduced to a system of Fredholm integral equations. Graphical display of the results are included.

The study of the torsional oscillations of an elastic solid is important in several fields, e.g., soil mechanics[1], mechanical transmission line theory[2] and transmission of power through shafts with flange at the end as integral part of the shaft. The oscillations of the half-space, excited by a rigidly attached circular disc, were first considered by Sagoci[3] and an approximate treatment of both the oscillating half-space and stratum was subsequently given by Bycroft[4]. Recently, Collins[5] has formulated exactly both the problems as Fredholm integral equations of the second kind, utilizing methods developed by him[6] in scalar diffraction theory.

The purpose of this paper is to investigate the effect of an embedded flaw in the form of a penny-shaped crack in an elastic half-space subjected to torsional oscillation. By following the fairly well-known procedure[7], the problem is reduced to a Fredholm integral equation. Attention is also drawn to the calculation of the ration M/M_0, where M_0 is the moment required to produce the rotation when the half-space contains no flaw. In addition, the shear-stress boundary value is considered along with other quantities of interest to such problems. Numerical results have been illustrated graphically.

FORMULATION OF THE PROBLEM

Let us consider an elastic half-space occupying the region $z \geqslant -h$, whose boundary $z=-h$ is stress free, except for the circular portion $0 \leqslant r < b$ to which is cemented a rigid circular shaft of radius 'b'. It is supposed that a penny-shaped crack is present in the region $0 \leqslant r \leqslant a$, $z=0$ whose faces are stress-free. Further, it is considered that a twisting moment of magnitude M is applied to the shaft causing it to rotate through an angle α.

It can be easily shown that, if we use cylinderical co-ordinates (r, θ, z), the displacement vector has only one non-vanishing component $U_\theta\ (r, z)$, and the stress tensor has only two non-vanishing components $\Upsilon_{r\theta}\ (r, z)$ and $\Upsilon_{\theta z}\ (r, z)$. The stress-strain relations reduce to two simple equations

$$\Upsilon_{r\theta} = \mu\, r \frac{\partial}{\partial r} \left(r^{-1} U_\theta \right), \tag{1}$$

$$\Upsilon_{\theta z} = \mu \frac{\partial U_\theta}{\delta z}, \tag{2}$$

where μ is shear modulus of the material. Since the problem is axisymmetric, the displacement vector U must satisfy

$$\frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} + \frac{{}^2 U_\theta}{\partial z^2} = 0 \tag{3}$$

The boundary conditions can be written in the form

$$U_\theta\ (r, -h) = \alpha r, \qquad 0 \leqslant r \leqslant b \tag{4}$$

$$Y_{\theta z}\left(r, -h\right) = 0, \qquad b < r < \infty \tag{5}$$

$$Y_{\theta z}\left(r, \overset{-}{0}\right) = Y_{\theta z}\left(r, \overset{+}{0}\right) = 0, \qquad 0 \leqslant r < a. \tag{6}$$

In addition to the above conditions, we have the following continuity conditions on $z = 0$

$$U_{\theta}\left(r, \overset{-}{0}\right) = U_{\theta}\left(r, \overset{+}{0}\right), \; a \leqslant r < \infty \tag{7}$$

$$Y_{\theta z}\left(r, \overset{-}{0}\right) = Y_{\theta z}\left(r, \overset{+}{0}\right), \; a \leqslant r < \infty \tag{8}$$

Now (6) and (8) show that

$$Y_{\theta z}\left(r, \overset{-}{0}\right) = Y_{\theta z}\left(r, \overset{+}{0}\right), \text{ for all r.} \tag{9}$$

SOLUTION OF THE PROBLEM

Let us take the solution of (3) as :

$$\int_0^\infty [\, A \cosh \xi z + B \sinh \xi z \,]\, J_1(\xi r)\, d\xi, \qquad -h < z < 0 \tag{10}$$

$$\int_0^\infty C e^{-\xi z} J_1(\xi r)\, d\xi, \qquad 0 < z < \infty \tag{11}$$

where A, B and C are functions of 'ξ' and are to be determined from the boundary conditions and continuity conditions just stated.

From (9) and (11), it follows that

$$C(\xi) = -B(\xi) \tag{12}$$

Now imposing the remaining boundary conditions, we see that $A(\xi)$ and $B(\xi)$ satisfy

$$\int_0^\infty [\, A(\xi) \cosh(\xi h) - B(\xi) \sinh(\xi h) \,]\, J_1(\xi r)\, d\xi = \alpha r, \; 0 \leqslant r \leqslant b, \tag{13}$$

$$\int_0^\infty \xi\, [\, A(\xi) \sinh(\xi h) - B(\xi) \cosh(\xi h)]\, J_1(\xi r)\, d\xi = 0, \; b < r < \infty, \tag{14}$$

$$\int_0^\infty \xi\, B(\xi)\, J_1(\xi r)\, d\xi = 0, \qquad 0 \leqslant r < a \tag{15}$$

$$\int_0^\infty [\, A(\xi) + B(\xi) \,]\, J_1(\xi r)\, d\xi = 0, \quad a < r < \infty \tag{16}$$

REDUCTION OF THE PROBLEM TO FREDHOLM INTEGRAL EQUATIONS

Let the trial solution be

$$A(\xi) \sinh(\xi h) - B(\xi) \cosh(\xi h) = \int_0^b m(t) \sin \xi t\, dt, \tag{17}$$

$$A(\xi) + B(\xi) = \int_0^a n(t) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) dt \tag{18}$$

With this choice of the unknown functions, we see that equations (14) and (16) are satisfied. Now solving (17) and (18) for A and B,
we get

$$A(\xi) = e^{-\xi h} \int_0^b m(t) \sin \xi t \, dt + e^{-\xi h} \cosh (\xi h) \int_0^a n(t) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) dt , \tag{19}$$

and

$$B(\xi) = -e^{-\xi h} \int_0^b m(t) \sin \xi t \, dt + e^{-\xi h}, \sinh (\xi h) \int_0^a n(t) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) dt \tag{20}$$

By inserting these values of A and B in (13), we get

$$\int_0^\infty \left[\int_0^b m(t) \sin \xi t \, dt + e^{-\xi h} \int_0^a n(t) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) dt \right] J_1(\xi r) \, d\xi = \alpha r, \; 0 \leqslant r \leqslant b$$

Changing the order of integration and using the results

$$\int_0^\infty J_1(\xi r) \sin \xi t \, d\xi = \frac{t \, H(r-t)}{r\sqrt{r^2 - t^2}} ,$$

$$J_1(\xi r) = \frac{2}{\pi r} \int_0^r \frac{x \sin \xi x}{\sqrt{r^2 - x^2}} .$$

we get

$$\int_0^r \frac{x}{\sqrt{r^2 - x^2}} \left[m(t) + \frac{2}{\pi} \int_0^a n(t) \int_0^\infty e^{-\xi h} \sin \xi h \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) d\xi \, dt \right] dx = \alpha r^2 \, (0 \leqslant r \leqslant b) \tag{21}$$

which is an Abel type of equation whose solution is

$$m(x) + \int_0^a K(x, t) \, n(t) \, dt = \frac{4\alpha}{\pi} x , \; (0 \leqslant x \leqslant b) \tag{22}$$

where

$$K(x, t) = \frac{2}{\pi} \int_0^\infty e^{-\xi h} \sin \xi x \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) d\xi \tag{23}$$

The integral can be evaluated in closed form to give

$$K(x, t) = \frac{1}{2\pi t} \log \frac{h^2 + (x+t)^2}{h^2 + (x-t)^2} - \frac{1}{\pi} \left[\frac{x+t}{h^2 + (x+t)^2} + \frac{x-t}{h^2 + (x-t)^2} \right] \tag{24}$$

in which the singularity at $t=0$ is illusory. That $K(x, t)$ is continuous for all x and t follows from the easily established uniform convergence of its defining integral for all x, t. In fact $K(x, 0) = 0$.

Now integrating by parts the second part of (20), we get

$$2 \xi B(\xi) = \int_0^a \left\{ \frac{n(t)}{t} + n'(t) \right\} \sin \xi t \, dt - n(a) \sin \xi a -$$

$$- \xi \; e^{-2\xi h} \int_0^a n(t) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) dt -$$

$$- 2\xi \; e^{-2\xi h} \int_0^b m(t) \sin \xi t \; dt \, . \tag{25}$$

Putting this in (15), and interchanging the order of integration and making use of the relation

$$\int_0^\infty J_1(\xi r) \sin \xi t \; d\xi = \frac{t \, H(t-r)}{r \sqrt{r^2 - t^2}} \, ,$$

$$J_1(\xi r) = \frac{2}{\pi r} \int_0^r \frac{x \sin \xi x}{\sqrt{r^2 - x^2}}$$

we obtain the Abel equation

$$\int_0^r \frac{1}{\sqrt{r^2 - y^2}} \left[\frac{d}{dy} \{ y \, n(y) \} - \frac{2}{\pi} \int_0^a n(t) \, dt \int_0^\infty \xi \, e^{-2\xi h} \; y \sin \xi y \, . \right.$$

$$\left. \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) d\xi - \frac{4}{\pi} \int_0^b m(t) \, dt \int_0^\infty \xi \, e^{-\xi h} \; y\xi . \sin(\xi y) \sin(\xi t) \, d\xi \right] dy \; (0 \leqq r < a) \tag{26}$$

On integrating with respect to y from 0 to x, with x in interval $(0, a)$ and then dividing by x, we finally obtain

$$n(x) - \int_0^a \mathrm{L}(x, t) \, n(t) \, dt = 2 \int_0^b K(t, x) \, m(t) \, dt, \; 0 \leqslant x < a \tag{27}$$

where K (t, x) as the notation implies the result of interchanging x and t in $K(x, t)$ given by (23) and

$$\mathrm{L}(x, t) = \frac{2}{\pi} \int_0^\infty e^{-2\xi h} \left(\frac{\sin \xi x}{\xi x} - \cos \xi x \right) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) d\xi \tag{28}$$

This integral, which is also uniformly convergent for all x and t, can likewise be evaluated in closed form. Hence the kernel L is continuous and is given by

$$\mathrm{L}(x, t) = \frac{2h}{\pi} \left[\frac{1}{4h^2 + (x+t)^2} + \frac{1}{4h^2 + (x-t)^2} - \frac{1}{2xt} \log \frac{4h^2 + (x+t)^2}{4h^2 + (x-t)^2} \right] \tag{29}$$

Since angle α is not given, we must add to the equations (22) and (30) the equation which expresses the fact that the external moment applied to the shaft is M. It is easily seen that the moment exerted on the shaft by the half-space is

$$2\pi \int_0^b r^2 \, Y_{\theta z}(r, -h) \, dr \tag{30}$$

the integrand of which can be expressed in terms of $m(t)$, and then equilibrium of the shaft requires that

$$\int_0^b t \, m(t) \, dt = \frac{M}{4 \pi \mu} \tag{31}$$

QUANTITIES OF PHYSICAL INTEREST

We now compute some quantities which are of great importance

Shearing Stress at $z = -h$

To compute the boundary value of $\Upsilon_{\theta z}(r, -h)$ for $0 \leqslant r < b$ put the value of U_θ from (11) in (2) to get

$$\Upsilon_{\theta z}(r, -h) = \mu \int_0^\infty [-A(\xi)\sinh(\xi h) + B(\xi)\cosh(\xi h)]\, \xi . J_1(\xi r)\, d\xi ,$$

which on substituting from (17) and using the relation $J'_0(x) = -J_1(x)$ becomes

$$\mu \frac{d}{dr} \int_0^b m(t)\, dt \int_0^\infty J_0(\xi r) \sin \xi t \, d\xi \tag{32}$$

If we use

$$\int_0^\infty J_0(\xi r) \sin \xi t \, d\xi = \begin{cases} 0, & t < r \\ (t^2 - r^2)^{-\frac{1}{2}}, & t > r \end{cases} \tag{33}$$

and perform the indicated differentiation, we obtain

$$\Upsilon_{\theta z}(r, -h) = -\mu \left[\frac{b\, m(t)}{r\sqrt{b^2 - r^2}} - \frac{1}{r} \int_0^b \frac{t\, m'(t)\, dt}{\sqrt{t^2 - r^2}} \right], \ 0 \leqslant r < b \tag{34}$$

It is not difficult to show from (34) that $\Upsilon_{\theta z}(r, -h)$ is $O(r)$ as $r \to 0^+$ and that the integral remains bounded as $r \to b^-$. Hence we get square root singularity at $r = b$; and, the constant $m(b)$ or its equivalent $m(1)$ from (40), is a measure of the strength of the singularity at the rim of the shaft.

Shearing Stress at $z = 0$

To compute $\Upsilon_{\theta z}(r, 0)$ for $r > a$, we first note from (2) and (11) that

$$\Upsilon'_{\theta z}(r, o) = \mu \int_0^\infty \xi B(\xi) J_1(\xi r)\, d\xi, \tag{35}$$

Now using (20), we obtain

$$\Upsilon_{\theta z}(r, 0) = -\tfrac{1}{2}\mu \frac{a\, n(a)}{r\sqrt{r^2 - a^2}} + \tfrac{1}{2}\mu \int_0^a \left\{ \frac{n(t)}{t} + n'(t) \right\} dt . \int_0^\infty J_1(\xi r) \sin \xi t \, d\xi \, -$$

$$-\tfrac{1}{2}\mu \int_0^a n(t) \int_0^\infty \xi e^{-2\xi h} J_1(\xi r) \left(\frac{\sin \xi t}{\xi t} - \cos \xi t \right) d\xi - \mu \int_0^b m(t)\, dt \, .$$

$$. \int_0^\infty \xi e^{-2\xi h} \sin \xi t \left(\frac{\sin \xi r}{\xi r} - \cos \xi r \right) d\xi \, .$$

$$= -\tfrac{1}{2} \quad \mu \quad \frac{a \; n \; (a)}{r \sqrt{r^2 - a^2}} + U \; (r) \tag{36}$$

where

$$U \; (r) = \tfrac{1}{2} \; \mu \int_0^\infty \left\{ \frac{n \; (t)}{t} + n' \; (t) \right\} dt. \int_0^\infty J_1 \; (\xi r) \sin \xi t \;\; d\xi - \tfrac{1}{2} \; \mu \int_0^a n \; (t) \; .$$

$$\cdot \int_0^\infty \xi \; e^{-2 \xi h} \; J_1 \;\; (\xi r) \left(\frac{\sin \;\; \xi t}{\xi t} - \cos \; \xi t \right) d\xi -$$

$$- \mu \int_0^b m \; (t) \; dt \int_0^\infty \xi \; e^{-2 \xi h} \; \sin \; \xi \; t \; \left(\frac{\sin \;\; \xi r}{\xi r} - \cos \; \xi r \right) d\xi \quad . \tag{37}$$

It can be easily shown that these remain bounded as $r \rightarrow \overset{+}{a}$. We conclude as before that $n \; (a)$ is a measure of the strength of the singularity around the periphery of the crack.

Torque applied to produce the given Boundary Conditions

The torque M, which must be applied to produce the prescribed boundary conditions, is given by the equation :

$$T = - \; 2 \; \pi \int_0^b r^2 \; Y_{\theta z} \; (r, - h \;) \; dr \tag{38}$$

On substituting from (2) and (11) and making use of the result

$$\int_0^b r^2 \; J_1 \; (\xi r) \; dr = \frac{b^2}{\xi} \; J_2 \; (b\xi)$$

we find that

$$T = 2 \; \pi \; \mu \; b^2 \int_0^\infty [\; A \; (\xi) \sin \; \xi h - B \; (\xi) \cosh \;\; \xi h \;] \; J_2 \; (\xi b) \; d\xi$$

$$= 2 \; \pi \; \mu \; b^2 \int_0^b m \; (t) \; dt \int_0^\infty \sin \;\; \xi t \; J_2 \;\; (\xi b) \;\; d\xi$$

and recalling that

$$\int_0^\infty J_2 \; (\xi b) \sin \;\; (\xi t) \; d\xi = \frac{2t}{b^2} \; (\; 0 \; \leqslant t \leqslant b \;)$$

we obtain the equation

$$T = 4\pi \;\; \mu \int_0^b t \; m \; (t) \; dt \tag{39}$$

NUMERICAL RESULTS

For numerical purposes it is convenient to write the system (22), (27) and (29) in the dimensionless form. Hence, if we set

$$\phi \; (u) \;\; = \frac{\pi \; m \; (bu)}{4 \; b \; \alpha} \quad \text{and} \quad \psi \; (u) \;\; = \frac{\pi \; n \; (au)}{4 \; a \; \alpha} \tag{40}$$

we obtain

$$\phi(u) + \int_0^1 R(u, v)\, \psi(v)\, dv = u, \qquad 0 \leqslant u \leqslant 1 \tag{41}$$

$$\psi(u) - \int_0^1 S(u, v)\, \psi(v)\, dv - \int_0^1 T(u, v)\, \phi(v)\, dv = 0, \qquad 0 \leqslant u \leqslant 1 \tag{42}$$

with

$$R(u, v) = \frac{a^2}{b} K(bu, av)$$

$$S(u, v) = a\ L\ (au, av)$$

$$T(u, v) = \frac{2\, b^2}{a} K(bv, au)$$

The kernels R, S, T can be written in explicit form as :

$$R(u, v) = \frac{\beta}{2\pi v} \log \frac{\gamma^2 + (u + \beta v)^2}{\gamma^2 + (u - \beta v)^2} - \frac{\beta^2}{\pi}\left[\frac{u + \beta v}{\gamma^2 + (u + \beta v)^2} + \frac{u - \beta v}{\gamma^2 + (u - \beta v)^2}\right] \tag{43}$$

$$S(u, v) = \frac{2\ \beta\gamma}{\pi}\left[\frac{1}{4\,\gamma^2 + \beta^2 (u + v)^2} + \frac{1}{4\,\gamma^2 + \beta^2 (u + v)^2}\right] - \frac{\beta\gamma}{\pi u v} \log \frac{4\gamma^2 + \beta^2 (u + u)^2}{4\gamma^2 + \beta^2 (u - v)^2} \tag{44}$$

$$T(u, v) = \frac{1}{\pi\,\beta^2 u} \log \frac{\gamma^2 + (v + \beta u)^2}{\gamma^2 + (v - \beta u)^2} - \frac{2}{\pi\beta}\left[\frac{v + \beta u}{\gamma^2 + (v + \beta u)^2} + \frac{v - \beta u}{\gamma^2 + (v - \beta u)^2}\right] \tag{45}$$

where

$$\beta = \frac{a}{b} \text{ and } \gamma = \frac{h}{b}$$

The numerical treatment of the system which governs the problem was the usual one i.e. the system was approximated by sets of linear equations. The basic interval (0, 1) was first partitioned into 10 equal sub-intervals and that the trapezoidal rule was used in the treatment of the integrals. The linear equations were solved for the ten functional values each of ϕ (t) and ψ (t) for $t = 0 \cdot 1, \ldots\ldots\ldots\ldots;$ $1 \cdot 0$ [$\Phi(0) = \Psi(0) = 0$ is obvious from the integral equations themselves.]

The relation

$$3 \int_0^1 t\ \ \phi\ \ (t)\, dt = \frac{M}{M_0}$$

which is present in the system, was not used in the solution; instead, it was used to evaluate the ratio M/M_0 after the functional values of ϕ were obtained.

To assess the accuracy of the solutions, the calculations were repeated using 20 sub-intervals and Simpson's rule. The second set of results was practically indistinguishable from the first. The results in question are shown in Figs. 1—3 which respectively show the variations of ϕ (1), ψ (1), M/M_0, for $\gamma = 0 \cdot 2$, $0 \cdot 45$, $0 \cdot 75$ and $1 \cdot 0$.

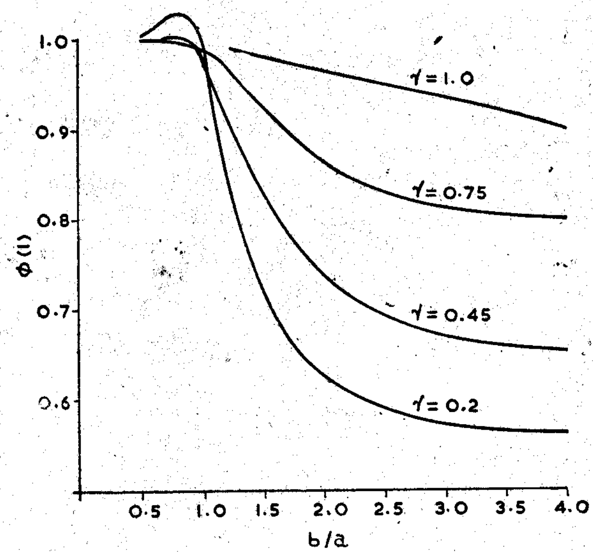

Fig. 1—Effect of flaw on applied torque.

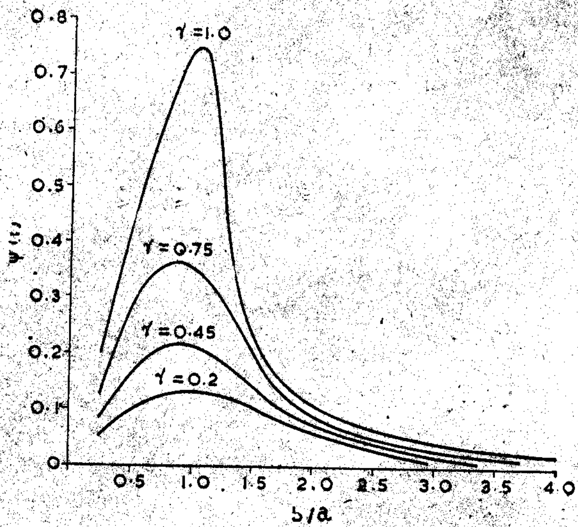

Fig. 2—Stress intensity factor at rim of shaft.

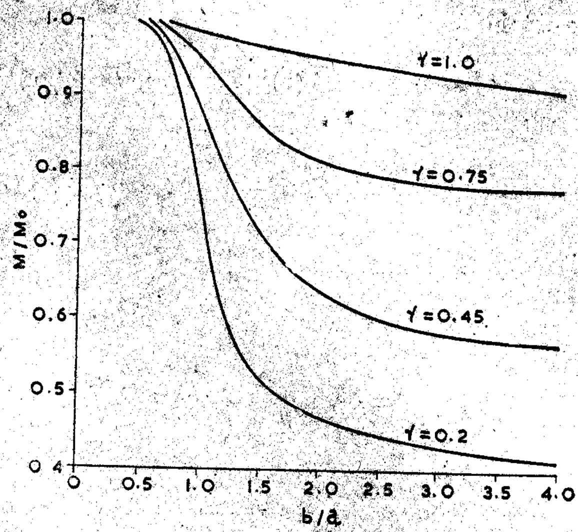

Fig. 3—Stress intensity factor at rim of flaw.

ACKNOWLEDGEMENT

I am thankful to the referee for certain useful suggestions.

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