SOLUTION OF A FIRST ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

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A first order linear partial differential equation in two independent variables, involving a partial differential operator with constant co-efficients, can be solved by developing and using the properties of the operator in much the same way as the properties of a total differential operator are used in the solution of a linear total differential equation, with the difference that a certain consistency relation has to be satisfied by the boundary condition in the former case.

The first order linear partial differential equation occurs in certain problems on heat convectio~, viz., in circulating fuel reactors, whereas liquid fuel flows a lorg channel with a given velocity[1]. In such cases one generally resorts to Fourier Transform techniques which are not always convenient to use, particularly when there are terms present which are functions of the independent variables.

In this paper it is proposed to develop a simple method for inverting a general linear partial differential operator, which may contain a function of the independent variables, for obtaining a solution satisfying a linear boundary condition and to express it in the form of a standard integral formula.

The first order linear partial differential equation in two independent variables can be written as

$$\left[\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + p(x, y) \right] V(x, y) = \phi(x, y) \tag{1}$$

where $p(x,y)$, $\phi(x, y)$ are sufficiently well-behaved functions if x,y. This equation is analogous to the ordinary linear differential equation.

$$[d/dx + p(x)]\, V(x) = \phi(x) \tag{2}$$

The solution of (2), for a given value of $V(0)$, can be written in the form of definite integral as

$$V(x) = e^{-\int_0^x p(s)\, ds} \left[V(o) + \int_0^x e^{\int_0^s p(t)\, dt} \phi(s)\, ds \right]$$

$$= e^{-\int_0^x p(s)\, ds}\, V(o) + \int_0^x e^{-\int_0^s p(t)\, dt} \phi(x-s)\, ds \tag{3}$$

In this note it is proposed to seek a solution of (1) in the form of a definite integral analogous to (3), satisfying a given boundary condition. In what follows, the symbol θ will be used for the partial operator

$$\left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} \right)$$

Solution :—Let the boundary condition imposed on (1) be

$$[V(x, y)]_{l_1 x + m_1 y = o} = F(l_2 x + m_2 y) \tag{4}$$

where (l_i, m_i) $(i=1,2)$ are constants. In order to find the solution of (1) subject to the boundary condition (4), use is made of the following identity

$$\theta \int_0^{l_1 x + m_1 y} \Psi(x - \lambda s, y - \mu s) = \Psi(x, y) \quad (5) \qquad \text{if } l_1 \lambda + m_1 \mu = 1. \tag{6}$$

which can easily be proved.

On defining the inverse operator θ^{-1} in the usual way, i.e., $\theta^{-1}\, \theta\, P(x,y) = P(x,y)$. Hence from (5) can be written as

$$\theta^{-1} \Psi(x, y) = \int_0^{l_1 x + m_1 y} \Psi(x - \lambda s, y - \mu s)\, ds \tag{7}$$

Equation (7) gives the interpretation of θ^{-1} in the form of a particular integral, and (6) is a consistency relation that has to be satisfied by the boundary condition.

Thus by (5), it is easily shown that

$$\theta\, e^{\int_0^{l_1x+m_1y} p\,(x-\lambda s,\, y-\mu s\,)\, ds}\, V\,(x, y\,) = e^{\int_0^{l_1x+m_1y} p\,(x-\lambda,\, y-\mu s\,)\, ds}\,[\,\theta + p\,(x, y\,)\,]\, V\,(x, y\,) \qquad (8)$$

From (1) & (7) this can be written as

$$[\,\theta + p\,(x, y\,)\,]^{-1}\, \phi\,(x, y\,) = e^{-\int_0^{l_1x+m_1y} p\,(x-\lambda s,\, y-\mu s\,)\, ds}\, \theta^{-1}\, e^{\int_0^{l_1x+m_1y} p\,(x-\lambda s,\, y-\mu s\,)\, ds}\, \phi\,(x, y\,) \quad (9)$$

$$= e^{-\int_0^{l_1x+m_1y} p\,(x-\lambda s,\, y-\mu s)\, ds} \int_0^{l_1x+m_1y} e^{\int_0^{l_1(x-\lambda s)+m_1(y-\mu s)} \phi\,(x-\lambda\,[\,s+t\,],\, y-\mu\,[\,s+t\,]\,)\, dt}\, \phi\,(\,x-\lambda s,\, y-\mu s\,)\, ds$$

$$= \int_0^{l_1x+m_1y} e^{-\int_0^{s} p\,(x-\lambda t,\, y-\mu t\,)\, dt}\, \phi\,(\,x-\lambda s,\, y-\mu s\,)\, ds \qquad (10)$$

by making use of (6) and simplifying; (10) gives the particular integral of the differential equation (1) which vanishes on the line $l_1x + m_1\, y = 0$.

To find the complimentary function of (1) which satisfies the given boundary condition, the transformation is made

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\triangle} \begin{bmatrix} m_2 & -m_1 \\ -l_2 & +l_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \text{ where } \triangle = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix},$$

in the function $F\,(\,K\,\{\,\lambda\, y - \mu\, x\,\}\,)$, which satisfies the homogeneous equation

$$\theta\, F\,(\,K\,\{\,\lambda\, y - \mu\, x\,\}\,) \text{ i.e., } \theta^{-1}\, 0 = F\,(\,K\,\{\,\lambda\, y - \mu\, x\,\}\,) \qquad (11)$$

It is found that, for $K = \triangle$ and on making use of (6)

$$\Big[F\,(\,K\,\{\,\lambda\, y - \mu\, x\,\}\,)\Big]_{l_1 x + m_1 y = 0} = F\,(v) \qquad (12)$$

Thus, writing 0 for $\phi\,(x, y)$ in (9), we get

$$[\,\theta + p\,(\,x,y\,)\,]^{-1} . \,0 = e^{-\int_0^{l_1 x + m_1 y} p\,(\,x-\lambda\, s\,,\, y-\mu\, s\,)\, ds}\, F\,(\,\triangle\,\{\,\lambda\, y - \mu\, x\,\}\,) \qquad (13)$$

Thus the complete solution of (1), satisfying boundary condition (4), is given by

$$V\,(x,y) = e^{-\int_0^{l_1 x+m_1 y} p\,(x-\lambda s,\, y-\mu s\,)\, ds}\, F\,(\triangle\{\,\lambda y-\mu x\,\}\,) + \int_0^{l_1 x+m_1 y} e^{-\int_0^{s} p\,(x-\lambda t,\, y-\mu t\,)\, dt}\, \phi\,(x-\lambda s,\, y-\mu s\,)\, ds \qquad (14)$$

which is seen to be analogous to solution (3) of the ordinary differential equation (2).

REFERENCE

1. MEGHREBLIAN, R.V. & HOLMES, D.K., "Reactor Analysis" (McGraw-Hill Book Co., Inc., New York), 1960, Chapter 9.