SOLUTION OF EQUATIONS OF INTERNAL BALLISTICS FOR THE COMPOSITE CHARGE USING LAGRANGE DENSITY APPROXIMATION

D.K. NARVILKAR

Armament Research & Development Establishment, Pune-411021

(*Received* 19 *September* 1978)

In the present paper, the equations of internal ballistics of composite charge consisting of N component charges with quadratic form function are solved. Lagrange density approximation and hydrodynamic flow behaviour, have been assumed and the solutions are obtained for the composite charge for these assumptions.

Methods based on the conventional density function C/Ax have been given by Corner[1], Hunt-Hind and Clemmow[2]. Clemmow has discussed the solutions of two composite charges of the same composition but of different shapes and sizes. Corner has reduced this problem to that of a single charge.

To consider gradual burning, Chugh[3,4] has suggested a new density function Cz/Ax. The theory has been extended for composite charges by Prasad[5,6]. Kapur[7,8&9], Venkatesan and Patni[10], Aggarwal[11], Gupta[12] and Tawakaley[13] have discussed the problem of composite charges under different conditions. A better approximation to the density of the propellant gases, viz , $\rho = \dfrac{Cz}{\left(Ax - \dfrac{C}{2\delta}\right)}$ has been given by Aggarwal, Modi and Varma[14].

Recently Narvilkar[15] has discussed lagrange approximation, $\rho = \dfrac{Cz}{\left[K_0 + Ax - \dfrac{C(1-z)}{\delta}\right]}$ to the density of the combustion products for single charge. In this paper, this density function has been used to evaluate the internal ballistic parameters for a composite charge.

BASIC EQUATIONS

Let the composite charge consists of N component charges and the ratio of specific heats is the same for each component. Subscripts i, b, s, m refers to the i^{th} component charge, conditions at the breech, shot base and mean values respectively of the parameter.

The well-known equations for the form function coefficient and combustion are

$$z_i = (1-f_i)\,(1+\theta_i f_i) \tag{1}$$

$$\frac{D_i}{\beta_i}\,\frac{df_i}{dt} = P_b, \tag{2}$$

and the energy equation is given by

$$\sum_{i=1}^{N} F_i\,C_i\,z_i = P_m\left[K_0 + Ax - \sum_{i=1}^{N} C_i\,z_i\,b_i - \sum_{i=1}^{N} \frac{C_i}{\delta_i}\,(1-z_i)\right] + (\gamma-1)\,W \tag{3}$$

Where W accounts for the work done by the reaction products in providing kinetic energy to the shot, propellant gases as well as the dissipation in overcoming bore resistance and heat transfer to the gun barrel. Equation for the constrained motion of the projectile within the barrel is

$$1{\cdot}05\,w\,\frac{dv}{dt} = AP_s \tag{4}$$

HYDRODYNAMIC FLOW BEHIND THE MOVING SHOT

The motion of the evolving combustion products behind the shot can be described by considering the equation of continuity and equation of conservation of momentum.

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial y} = \frac{\sum_{i=1}^{N} C_i \frac{dz_i}{dt}}{A\,(x+l) + \sum_{i=1}^{N} \frac{C_i z_i}{\delta_i}} \tag{5}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} \tag{6}$$

Here $u\,(y, t)$ is the gas velocity at a distance $y\,(y_b \leqslant y \leqslant y_s)$ at any instant t after the movement of the shot from the breech. The term, $\dfrac{\sum_{i=1}^{N} C_i \frac{dz_i}{dt}}{A\,(x+l) + \sum_{i=1}^{N} \frac{C_i z_i}{\delta_i}}$ denotes the mass rate per unit volume which is added to the gaseous products at any time and thus is the source function.

To make the equations dimensionless the following transformations are used

$$\left.\begin{aligned}
&\xi = 1 + \frac{x}{l}\,, \quad Al = K_0 - \sum_{i=1}^{N} \frac{C_i}{\delta_i} \\
&V_0 = \frac{\beta_N F_N C_N}{A\, D_N}\,, \quad \zeta = \frac{PAl}{F_N C_N}\,, \quad \zeta_b = \frac{P_b\, Al}{F_N C_N} \\
&\zeta_s = \frac{P_s\, Al}{F_N C_N}\,, \quad \zeta_m = \frac{P_m\, Al}{F_N C_N}\,, \quad \tau = \frac{V_0 t}{l} \\
&\eta = \frac{v}{V_0}\,, \quad Y = y/l, \quad U = \frac{v}{V_0}\,.
\end{aligned}\right\} \tag{7}$$

A mean density (called Lagrange density approximation)

$$\rho = \frac{\sum_{i=1}^{N} C_i z_i}{\left[K_0 + Ax - \sum_{i=1}^{N} \frac{C_i (1 - z_i)}{\delta_i} \right]}$$

has been assumed to be constant throughout the barrel at a given instant. The solution for the non-dimensional velocity U and pressure ζ from (5) and (6) are

$$U = \frac{\eta\,(Y - Y_b) + \frac{1}{Al}(Y - Y_s) \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau}}{\left(\xi + \frac{1}{Al} \sum_{i=1}^{N} \frac{C_i z_i}{\delta_i} \right)} \tag{8}$$

and .

$$\zeta = \zeta_s - \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{2 M_1 \left(\xi + \frac{1}{Al} \sum_{i=1}^{N} \frac{C_i z_i}{\delta_i} \right)} \cdot \left[\frac{2(Y-Y_s)\left(\frac{d\eta}{d\tau} + \frac{1}{Al} \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2} \right)}{\left(\xi + \frac{1}{Al} \sum_{i=1}^{N} \frac{C_i z_i}{\delta_i} \right)} + \right.$$

$$\left. + (Y-Y_s)^2 \frac{d\eta}{d\tau} \right] \tag{9}$$

$$\text{where } \epsilon = \frac{1}{1{\cdot}05\, w} \quad \text{and} \quad M_1 = \frac{F_N C_N}{1{\cdot}05\, w\, V_0^2}$$

It can be easily varified that $(Y_b - Y_s) = -\left(\xi + \frac{1}{Al} \sum_{i=1}^{N} \frac{C_i z_i}{\delta_i} \right)$, and then equation (9) at $Y = Y_b$ and $\zeta = \zeta_b$ reduces to

$$\zeta_b = \zeta_s + \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{2M_1} \cdot \left[\frac{d\eta}{dr} - \frac{\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2}}{Al} \right]. \tag{10}$$

The mean non-dimensional pressure defined as $\zeta_m = \frac{1}{(Y_s - Y_b)} \int_{Y_b}^{Y_s} \zeta dY$ can then be calculated using equation (9). It works out to be

$$\zeta_m = \zeta_s + \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{6 M_1} \cdot \left[2 \frac{d\eta}{d\tau} - \frac{\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2}}{Al} \right]. \tag{11}$$

and the kinetic energy of the propellant gases given by

$$E_p = \frac{1}{2} \int_{yb}^{ys} \rho v^2 A dy$$

equals to

$$E_p = \frac{V_0^2 \sum_{i=1}^{N} C_i z_i}{6} \cdot \left[\eta^2 + \left(\frac{1}{Al} \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau} \right)^2 - \frac{\eta}{Al} \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau} \right]. \tag{12}$$

Kinetic energy of the shot is $\frac{1}{2} wv^2$ and frictional losses due to bore resistance can be assumed equivalent to (0.05×K.E. of the shot). Thus the total contents of the mechanical work done by the system are

$$W = \frac{1.05\, w}{2}\, v^2 + \frac{V_0^2 \sum_{i=1}^{N} C_i z_i}{6} \cdot \left[\eta^2 + \left(\frac{1}{Al} \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau} \right)^2 - \frac{\eta}{Al} \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau} \right] + E_h$$

Where E_h is the energy losses due to heat transfer taken as

$$E_h = K_H \cdot \frac{1.05\, w V_0^2}{2}\, \eta^2$$

Let

$$(\bar{\gamma} - 1) = (\gamma - 1)(1 + K_H) .$$

Then energy equation (3) takes the form

$$\frac{\sum_{i=1}^{N} F_i C_i z_i}{F_N C_N} = \zeta_m \left[\xi - \frac{1}{Al} \sum_{i=1}^{N} C_i z_i \left(b_i - \frac{1}{\delta_i} \right) \right] + \frac{(\bar{\gamma} - 1)}{2M_1} \eta^2 +$$

$$+ \frac{\epsilon\, (\bar{\gamma} - 1)}{6M_1 (1 + K_H)} \left\{ \eta^2 \sum_{i=1}^{N} C_i z_i + \left(\sum_{i=1}^{N} C_i z_i \right) \times \right.$$

$$\left. \times \frac{1}{Al} \cdot \left(\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau} \right) \cdot \left[\frac{1}{Al} \sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{dz_i}{d\tau} - \eta \right) \right\} \tag{13}$$

In non-dimensional form equation of the motion of the shot, viz : equation (4) is

$$\frac{d\eta}{d\tau} = M_1 \zeta_s \tag{14}$$

where

$$\frac{d\xi}{d\tau} = \eta \tag{15}$$

By taking η as independent variable the above equations of internal ballistics transforms to

$$\frac{df_i}{d\eta} = - \frac{1}{M_1} \cdot \left(\frac{\beta_i}{D_i} \right) \frac{F_N D_N}{A V_0} \cdot \left(\frac{\zeta_b}{\zeta_s} \right) \tag{16}$$

$$\frac{d\xi}{d\eta} = \frac{\eta}{M_1 \zeta_s} \tag{17}$$

$$\zeta_b = \zeta_s \cdot \left(1 + \frac{\epsilon}{2} \sum_{i=1}^{N} C_i z_i \right) - \frac{\epsilon \left(\sum_{i=1}^{N} C_i z_i \right) \left(\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2} \right)}{2 M_1 Al} \tag{18}$$

and

$$\zeta_m = \zeta_s \cdot \left(1 + \frac{\epsilon}{3} \sum_{i=1}^{N} C_i z_i \right) - \frac{\epsilon \left(\sum_{i=1}^{N} C_i z_i \right) \left(\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2} \right)}{6 M_1 Al} . \tag{19}$$

RELATIONSHIP BETWEEN THE WEB-FRACTIONS REMAINING TO BE BURNT

From equation (2), we have

$$f_r = 1 + \frac{\beta_r}{D_r} \cdot \frac{D_N}{\beta_N} \cdot (f_N - 1) \tag{20}$$

where $r = 1, 2, \ldots\ldots\ldots\ldots, (N-1)$

Writing $\alpha_r = [\,(D_r/\beta_r)/(D_N/\beta_N)\,]$

where $r = 1, 2, \ldots\ldots\ldots\ldots, (N-1)$

equation (20) is written as $f_N = 1 - \alpha_r + \alpha_r f_r$

and since $\alpha_r f_r \geqslant 0$ for all $r < N$, thus we have

$f_N > 1 - \alpha_r$ before the charge of r^{th} size is burnt (21)

and thereafter

$$f_N < 1 - \alpha_r \tag{22}$$

From equations (1) and (20), we have

$$Z_r = \frac{(1 - f_N)}{\alpha_r} \cdot \left[\, 1 + \theta_r - \frac{\theta_r}{\alpha_r} \cdot (1 - f_N) \,\right] \tag{23}$$

where $r = 1, 2, \ldots\ldots\ldots\ldots, (N-1)$.

The equations (13) and from (16) to (23) form the complete set of internal ballistics system for composite charge.

Initial conditions—At $\tau = 0$ we have $\eta = 0$, $\xi = 1$ and $\zeta_s = \zeta_b = \zeta_m = \zeta_{ss}$, where ζ_{ss} is the non-dimensional shot start pressure. Using these initial conditions and relation (23) in equation (13) we can find the initial value of f_N. If the value of f_N so determined, is $< (1 - \alpha_1)$, it is understood that the shot does not start until the lowest size charge has burnt. As such z_i is put equal to 1 and f_N is determined again from equation (13) and initial conditions and equation (23). This time the value of f_N is again compared with $(1 - \alpha_2)$. If this value of f_N is $< (1 - \alpha_2)$ z_2 is put equal to 1 and so on till the calculated value of f_N is less than $(1 - \alpha_r)$. Then remaining values of z_i are calculated from equation (23) and (1).

Numerical solution—The system of equations (13 & 16—19) are approximated by ignoring the terms of second order derivatives of z_i and higher order terms.

$$\frac{df_N}{d\eta} = -\frac{1}{M_1} \cdot \left(1 + \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{2} \right) \tag{24}$$

$$\frac{d\xi}{d\eta} = \eta/(M_1 \zeta_s) \tag{25}$$

$$\zeta_b = \zeta_s \cdot \left(1 + \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{2} \right) \tag{26}$$

$$\zeta_m = \zeta_s \cdot \left(1 + \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{3} \right) \tag{27}$$

$$\frac{\sum_{i=1}^{N} F_i C_i z_i}{F_N C_N} = \zeta_m \cdot \left[\xi - \frac{1}{Al} \sum_{i=1}^{N} C_i z_i \left(b_i - \frac{1}{\delta_i} \right) \right] + \frac{(\bar{r} - 1)}{2 M_1} \eta^2 \tag{28}$$

These reduced equations are solved numerically using initial values of z_i, initial conditions and Runge-Kutta[16] algorithm. At each step of calculation the inequalities (21) and (22) are varified. Accordingly z_i values are substituted as 1 or determined by equation (23). Then using computed values of f_N and ξ_i the values of ζ_b, ζ_s, ζ_m are calculated from equations (26), (27) and (28). The evaluated values of ζ_b, z_i, ξ, ζ_s, ζ_m are used to find left out terms of equations (18) and (19) as

$$A' = \frac{\epsilon \left(\sum_{i=1} C_i z_i \right) \left(\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2} \right)}{2 M_1 Al} = \zeta_s \cdot \left[1 + \frac{\epsilon \sum_{i=1}^{N} C_i z_i}{2} \right] - \zeta_b$$

$$= \zeta_s \left\{ \left(1 + \frac{\epsilon}{2} \sum_{i=1}^{N} C_i z_i \right) - \frac{\epsilon A D_N}{V_0 \beta_N} \frac{dz_N}{d\eta} \frac{1}{[(1 + \theta_N^2) - 4 \theta_N z_N)]^{1/2}} \right\}$$

and

$$B' = \frac{\epsilon \left(\sum_{i=1}^{N} C_i z_i \right) \left(\sum_{i=1}^{N} \frac{C_i}{\delta_i} \frac{d^2 z_i}{d\tau^2} \right)}{6 M_1 Al} = \frac{A'}{3} \tag{29}$$

Then the values for ζ_b, ζ_m, ζ_s are corrected accordingly

$$(\zeta_m)_{\text{corrected}} = (\zeta_m)_{\text{calculated}} - B' \tag{30}$$

$$(\zeta_b)_{\text{corrected}} = \zeta_b \ {}_{\text{calculated}} \ {}^{-A'} \tag{31}$$

and

$$(\zeta_s)_{\text{corrected}} = \frac{[(\zeta_m)_{\text{calculated}} + B']}{\left(1 + \frac{\epsilon}{3} \sum_{i=1}^{N} C_i z_i \right)} \tag{32}$$

Density function of the reaction gases can be calculated from equation

$$\rho = \frac{\sum_{i=1}^{N} C_i z_i}{Al \left(\xi + \frac{1}{Al} \sum_{i=1}^{N} C_i z_i / \delta_i \right)} \tag{33}$$

After all burnt—Using above described numerical technique we can get values of ξ, ζ_b, ζ_m, ζ_s, ρ and η when all the propellant is burnt i.e. $f_N = 0$. Let the subscript *mb* represent conditions at total charge burnt. Solution after all burnt is given by

$$\zeta = \zeta_{mb} \left[\xi_{mb} - \frac{\sum\limits_{i=1}^{N} C_i \left(b_i - \frac{1}{\delta_i} \right)}{Al} \right]^{\left[1 + \frac{r-1}{\left(1 + \frac{\epsilon}{3} \sum\limits_{i=1}^{N} C_i \right)} \right]} \tag{34}$$

$$\zeta_s = \frac{\zeta_m}{\left(1 + \frac{\epsilon}{3} \sum\limits_{i=1}^{N} C_i \right)} \tag{35}$$

$$\zeta_b = \zeta_s \cdot \left(1 + \frac{\epsilon}{2} \sum_{i=1}^{N} C_i \right) \tag{36}$$

$$\eta^2 = \frac{2M_1}{(\bar{\zeta} - 1)} \cdot \left\{ \frac{\sum\limits_{i=1}^{N} F_i C_i}{F_N \; C_N} - \zeta_m \left[\xi_m - \frac{1}{Al} \sum_{i=1}^{N} C_i \left(b_i - \frac{1}{\delta_i} \right) \right] \right\} \tag{37}$$

and density of the combustion products after all burnt is given by

$$\rho = \frac{\sum\limits_{i=1}^{N} C_i}{Al \left[\xi + \frac{1}{Al} \sum\limits_{i=1}^{N} (C_i/\delta i) \right]} \tag{38}$$

DISCUSSION OF THE RESULTS AND CONCLUSIONS

Results obtained by using above numerical technique and Hunt-Hinds method, for experimental data given in Table 1, are presented in Table 2. The mean pressure distribution against the distance travelled

TABLE 1

EXPERIMENTAL DATA

Vol. of the chamber	120 cu in.	Shot travel	60 in.
Bore area	7.0543 sq in.	Shot start pressure	3.5 tsi.
Shell Wt	13.32 lb	K_H	0.1
		γ	1.25

PARTICULARS OF THE COMPOSITE CHARGE OF THREE COMPONENT CHARGES

Component Charge	Form coefficient	Rate of burning (in/sec)	Web size (in.)	Force constant (in-tons/lb)	Weight			covolume (cu in/lb)	Density (lb/cu in.)
					lb	oc	dr		
No. 1	1	0.75	0.018	1900		4	11½	27	0.06061
No. 2	—0.172	0.75	0.0322	1900		4	12	27	0.06061
No. 3	—0.172	0.75	0.0414	1900	1	11	13	27	0.06061

TABLE 2

RESULTS OBTAINED FROM HUNT-HIND'S METHOD AND PRESENT METHOD

Particulars	HUNT-HIND'S Method	Present Method
Muzzle velocity	1997 ft/sec	2023 ft/sec
Maximum Pressure	16.8 tsi	16.5 tsi
All burnt position	43.8 in.	44.5 in.

by the shot is shown in Fig. 1. Initially mean pressure is high in the Hunt-Hinds method than the new method but after all burnt it is low. Predicted maximum pressure using the new method is less than the Hunt-Hinds predicted maximum pressure. Density curves predicted in Fig. 2 shows that the density of the propellant gases, based in the Hunt-Hinds method is maximum initially and then decreases continuously. Density curve based on the present method shows that the density of the combustion products increases in the initial stage of the movement of the shot and then slowly decreases. All burnt position predicted by the Hunt-Hind's method occurs early than the predicted all burnt position of the present method. Muzzle velocity calculated by the new method is close to the Hunt-Hinds predicted muzzle velocity.

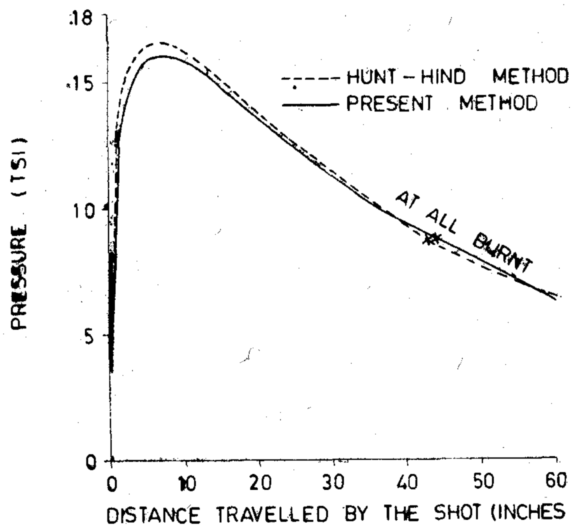

Fig. 1—Variation of the mean pressure with the distance travelled by the shot.

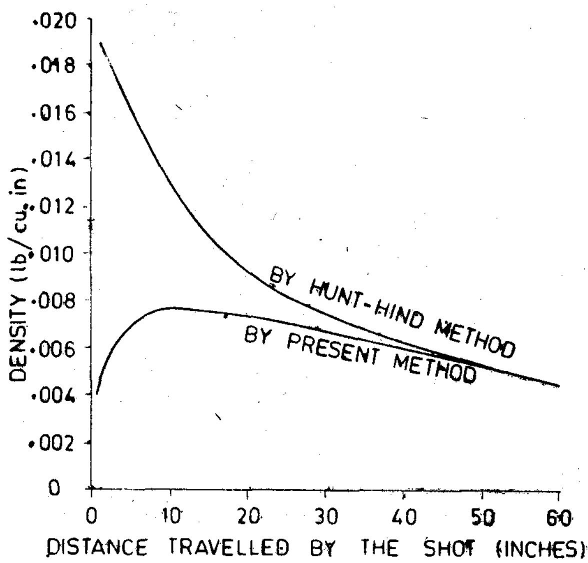

Fig. 2—Variation of the density of the propellant gas with the distance travelled by the shot.

The method presented here is very easy, provided a computer is used. The system of equations discussed above have been derived from the fundamental theory of hydrodynamics and Lagrange density function without any other approximations and assumptions. The above method gives results which matches closely with that of experimental values.

It may be pointed out that the technique presented is capable of accurately simulating gun cycle for any loading conditions.

ACKNOWLEDGEMENT

The author is grateful to Shri N. S. Venkatesan, Director, A.R.D.E , Pune, for his encouragement. The author wishes to express his gratitude to Dr. K. C. Sharma, Institute of Armament Technology, Pune, for his continuous guidance in the preparation of the paper.

REFERENCES

1. CORNER, J., 'Theory of Interior Ballistics of Guns' (John Wiley & Sons, New York), 1950, p. 192.
2. CLEMMOW, C.A., 'Internal Ballistics' (HMSO, London), 1951, p. 134.
3. CHUGH, O. P., Mem. *Artillery*, **41** (1967), 247.
4. CHUGH, O. P., *Mem. Artillery*, **43** (1969), 195.
5. PRASAD, G. R., *Mem. Artillery*, **44** (1970), 195.
6. PRASAD, G. R., *Mem. Artillery*, **43**, (1969), 793.
7. KAPUR, J. N., *Proc. Nat. Inst. Sci. India*, **23A** (1957), 16-39.
8. KAPUR, J. N., *Def. Sci., J.*, **6** (1957), 26-40.
9. KAPUR, J. N., *Proc Nat. Inst. Sci. India*, **28** (1962), 368-381.
10. VENKATESAN, N. S. & PATNI, G. C., *Def. Sci., J.*, **3**, (1953), 51-60.
11. AGGARWAL, S. P., *Proc. Nat. Inst. Sci. India*, **11A** (1958), 321-330.
12. GUPTA, M. C., *Def. Sci. J.*, **10** (1960), 12-23.
13. TAWAKLEY, V. B., *Def. Sci. J.*, **5**, (1955), 288-297.
14. AGGARWAL, S. P., MODI, J. K., VARMA, P. S., *Mem. Artillery*, **44** (1970), 415.
15. NARVILKAR, D. K., *Def. Sci., J.*, 4, (1977), 175-182.
16. SCARBOROUGH, J. B., 'Numerical Mathematical Analysis' (The John's Hopkins Press, London), (1964), 356.