PROPAGATION OF CYLINDRICAL AND SPHERICAL EXPLOSION WAVES IN AN EXPONENTIAL MEDIUM

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Solutions for a strong point or line explosion, in a medium whose density increases exponentially under low constant pressure, are obtained in this paper. The disturbance is headed by a shock surface and the total energy of the wave increases with time.

Propagation of planar shock waves, in a medium whose density follows an exponential law, has been studied by Raizer[1], Grover and Hardy[2] & Deb Ray[3]. The importance in such a medium, of cylindrical and spherical blast waves, to astrophysical problems, has been brought out by Grover & Hardy[2].

This paper presents a study of a strong point or line explosion in a gaseous medium under low uniform pressure and with a density that increases exponentially from the site of explosion. The disturbance is headed by a strong shock surface, spherical or cylindrical. Effects of viscosity and heat conduction have not been taken into account. The integrals to the hydrodynamical equations, involving quadratures, form a set of non-similarity solutions. Besides, it is interesting to find, as desirable, that the total energy of the wave increases with time. The strength of the shock at the head of the wave, although very large, remains constant throughout.

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The flow behind a cylindrical or a spherical shock surface are governed by the following equations of motion :

$$\frac{\partial E}{\partial t} + \frac{1}{r^{\nu}} \frac{\partial}{\partial r} (r^{\nu} u I) = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{\nu \rho u}{r} = 0 \tag{2}$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} = \frac{\gamma p}{\rho} \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} \right), \tag{3}$$

where
$$E = \tfrac{1}{2} \rho u^2 + p/\gamma\text{-}1 \tag{4}$$

and
$$I = \tfrac{1}{2} \rho u^2 + \gamma\, p/\gamma\text{-}1, \tag{5}$$

u, p, ρ represent the velocity, pressure and density of an element of gas at a radial distance r from the centre or the line of explosion ; γ is the ratio of specific heats and $\nu = 1$ or 2 for the cylindrical and spherical symmetry respectively.

If ahead of the shock, moving outwards with velocity V ($= \frac{dR}{dt}$), the undisturbed density be ρ_0 and the pressure, density and velocity just behind the shock be p_1, ρ_1 and u_1 respectively, then the conditions for a strong shock are

$$u_1 = \frac{2\ V}{\gamma + 1} \tag{6}$$

$$p_1 = \frac{2\ \rho_0\ V^2}{\gamma + 1} \tag{7}$$

$$\rho_1 = \rho_0 \frac{\gamma + 1}{\gamma - 1} \tag{8}$$

Let p_0 be a small constant and

$$\rho_0 = \rho^* e^{\beta R} \tag{9}$$

where R denotes the radius of the shock, ρ^* and β being suitable constants.

Let us seek solutions of the equations (1)–(3) in the form

$$u = \frac{1}{t} U(\eta) \tag{10}$$

$$p = t^{\mu-2} P(\eta) \tag{11}$$

$$\rho = t^{\mu} \Omega(\eta) \tag{12}$$

$$\eta = t e^{\lambda r} \tag{13}$$

where constants μ and λ are to be determined from the conditions of the problem. At the shock surface, the value of η is assumed constant. Hence

$$V = -\frac{1}{\lambda t} \tag{14}$$

In order that (14) may represent an outgoing shock surface $\lambda < 0$.

The solutions of equations (1)–(3) in the form (10)–(14) are compatible with the shock conditions, if

$$\mu = 2 \text{ and } \lambda = -\beta/2 \quad . \tag{15}$$

Since necessarily $\lambda < 0$, (15) shows that $\beta > 0$. In other words, the shock surface expands outwardly in an exponentially increasing medium. It can be easily verified that the strength of the shock under these conditions, remains constant.

From (14) and (15), we get

$$R = \frac{2}{\beta} \log \frac{t}{t_0} \quad , \tag{16}$$

t_0 being the duration of the al most instantaneous explosion.

SOLUTIONS OF THE EQUATIONS OF MOTION

The condition inside the wave is obtained from the solutions of equations (1)–(3). From (12), (14) and (15), we get

$$\frac{\partial \rho}{\partial t} = \frac{\mu}{t} \rho + \frac{1}{\lambda t} \frac{\partial \rho}{\partial r} \cdot \tag{17}$$

From (11), (14) and (15), we get

$$\frac{\partial p}{\partial t} = -V \frac{\partial p}{\partial r} \cdot \tag{18}$$

Equation (1) may be re-written in the form

$$\frac{dE}{dr^1} = \frac{1}{(r^1)^{\nu}} \frac{d}{dr'} \left(r'^{\nu} u' I \right), \tag{19}$$

which on integration, after using conditions (6)—(8), gives

$$\frac{E}{E_1} - u' \quad \frac{I}{E_1} = \nu \int_1^{r'} \frac{u'}{r'} \cdot \quad \frac{I}{E_1} \, dr' \quad . \tag{20}$$

Here, E_1 and I_1 represent respectively the values of E and I just inside the shock surface. Besides, we set

$$r' = \frac{r}{R} \quad \text{and} \quad u' = \frac{u}{V} \quad .$$

We can also express

$$\frac{E}{E_1} = \frac{(\gamma+1)^2}{\gamma} \cdot \frac{\rho}{\rho_1} \cdot u'^2 + \tfrac{1}{2} \cdot \frac{p}{p_1} \tag{21}$$

and

$$\frac{I}{E_1} = \frac{(\gamma+1)^2}{\gamma} \cdot \frac{\rho}{\rho_1} \cdot u'^2 + \tfrac{1}{2} \cdot \frac{\gamma p}{p_1} \quad . \tag{22}$$

Equation (2), by using (17), may be written as

$$\frac{1}{\rho} \cdot \frac{\partial \rho}{\partial r'} = \frac{\lambda \mu R}{u'-1} + \frac{1}{1-u'} \cdot \frac{\partial u'}{\partial r'} + \frac{1}{1-u'} \cdot \frac{\nu u'}{r'} \tag{23}$$

which on integration gives

$$\log \frac{\rho}{\rho_1} = 2 \log \frac{t}{t_0} \int_1^{r'} \frac{dr'}{1-u'} + \nu \int_1^{r'} \frac{dr'}{r'(1-u')}$$

$$- \log \frac{1-u'}{1-u_1'} - \nu \log r' \tag{24}$$

By using (17) and (18), equation (3) can be written as

$$\frac{1}{p} \cdot \frac{\partial p}{\partial r'} = \frac{\gamma}{\rho} \cdot \frac{\partial \rho}{\partial r'} + \frac{\gamma \beta R}{u'-1} \quad .$$

This on integration gives

$$\log \frac{p}{p_1} = \gamma \log \frac{\rho}{\rho'} - \gamma \left(2 \log \frac{t}{t_0} \right) \int_1^{r'} \frac{dr'}{1-u'} \quad . \tag{25}$$

Equations (20), (24) and (25), involving quadratures, give the solution of our problem, $\nu = 1$ and 2 representing the cylindrical and spherical case respectively. In the case $\nu = 0$, representing a plane explosion, the solutions reduce to simple algebraic integrals as has been studied by one of us[3].

The total energy of the wave is non-constant, and varies as the square or the cube of the shock radius accordingly as the explosion is of cylindrical or spherical symmetry.

NUMERICAL INTEGRATIONS

For numerical evaluation, we choose a particular value of γ. This enables us to obtain the values of u_1/V, $p_1/\rho_0 V^2$ and ρ_1/ρ_0 from equations (6)–(8). Next, a given instant of time corresponds to a definite value for t/t_0 which is necessary to get numerical results.

We can now proceed with integration from the shock front inwards. At the shock front $r' = 1$ and u_1/V is known. The next step consists in taking a set of neighbouring values of r' below 1, and choosing by method of trial and error the corresponding value of u', such that equations (20), (24) and (25) are satisfied. The quadratures are evaluated by Simpson's method, the mean value for u' being taken at the mid-interval. This may be carried out until the singularity of the solution $u' = 1$ is reached. This corresponds to the critical surface marking the expanding surface of the inner vacuous region.

Numerical integrations have been carried out for a spherical explosion. The results, calculated for only two particular cases A and B representing $\frac{t}{t_0} = 2$ and 10, and $\gamma = 1{\cdot}4$ are given in Tables 1 and 2.

TABLE 1

NUMERICAL INTEGRATIONS FOR SPHERICAL EXPLOSION

Case A : $\nu=2$, $\gamma=1{\cdot}4$, $t/t_0=2$.

r'	u'	ρ/ρ_1	p/p_1
1	·833	1	1
·975	·873	·736	·909
·950	·913	·520	·879
·925	·961	·285	·851
·915	·975	·192	·793
·905	·991	·094	·756

TABLE 2

NUMERICAL INTEGRATION FOR SPHERICAL EXPLOSION

Case B : $\nu=2$, $\gamma=1.4$, $t/t_0=10$.

r'	u'	ρ/ρ_1	p/p_1
1	·833	1	1
·975	·849	·412	·793
·950	·881	·160	·695
·925	·921	·048	·635
·900	·973	·004	·625

As is well-known, the conservation of mass within the shock surface is given by

$$\int_{r_1'}^{1} \frac{\rho}{\rho_1} \cdot r'^2 \, dr' = \frac{\gamma-1}{\gamma+1}\left[\frac{1}{\beta R} - \frac{2}{\beta^2 R^2} + \frac{2}{\beta^3 R^3}\left(1 - e^{-\beta R}\right)\right] \qquad (26)$$

In Case A, R.H.S. of (26) yields ·041,

whereas $\int_{\cdot 905}^{1} \frac{\rho}{\rho_1} \cdot r'^2 \, dr' = .0174$.

In Case B, R.H.S. of (26) equals ·0238

whereas $\int_{\cdot 9}^{1} \frac{\rho}{\rho_1} \cdot r'^2 \, dr' = .026$.

This small over-estimation in calculation of mass in each of the cases testifies that our numerical results are sufficiently accurate and our approach is practicable and useful. Besides, we conclude that in both the cases, the entire mass in the spherical blast wave forms a very thin shell near the outer boundary.

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