A NOTE ON THE LAMINAR HEAT TRANSFER IN AN ANNULUS

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An exact solution for the fluid temperature due to forced convective heat transfer in an annulus is obtained as a solution of the eigen value problem.

The annulus represents a common geometry employed in a variety of heat transfer systems ranging from simple heat exchangers to the most complicated nuclear reactors. In this note, a theoretical analysis for the forced convective heat transfer in an annulus is presented. The exact solution for the fluid temperature is determined in the most general form, as a solution of the eigen value problem.

BASIC EQUATION AND SOLUTION

Consider the flow of an incompressible viscous fluid in an annulus, taking the cylindrical polar system (r, ϕ, x) in which the axis of the cylinders is along the x-axis and r denoting radial distance. The walls of the annulus $r = a$ and $r = b$ are kept at uniform temperatures.

The local axial velocity for the fully developed laminar flow can be obtained as

$$u(r) = \frac{2\, U_m}{(s^2 + 1)\, lns - (s^2 - 1)} \left[(1 - r^2/a^2)\, lns + (s^2 - 1)\, ln\, r/a \right] \tag{1}$$

where U_m is the mean velocity and $s = b/a$.

Assuming that the fluid has constant properties and including the effect of axial conduction and viscous dissipation the energy equation is

$$u \frac{\partial T}{\partial x} = k \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2} \right) + \frac{\nu}{c} \left(\frac{\partial u}{\partial r} \right)^2 + \frac{q}{\rho c}, \tag{2}$$

where c is the specific heat, k the thermo-metric conductivity, ν the kinematic viscosity, $q/\rho c$ the uniform heat generation.

The Boundary conditions are

$$\left. \begin{aligned} T(x, a) &= T^{+}{}_{a} \\ T(x, b) &= T^{+}{}_{b} \end{aligned} \right\} \text{ for } x > 0 \tag{3}$$

$$\left. \begin{aligned} T(x, a) &= T^{-}{}_{a} \\ T(x, b) &= T^{-}{}_{b} \end{aligned} \right\} \text{ for } x < 0 \tag{4}$$

We now solve the energy equation as an eigen value problem.

Let T_1 denotes the solution of (2) for $x < 0$ and T_2 the solutions for $x > 0$. These two solutions are obtained separately and matched at $x = 0$ by using the following matching conditions :

$$\left. \begin{aligned} T_1 (0, r) &= T_2 (0, r) \\ \frac{\partial T_1 (0, r)}{\partial x} &= \frac{\partial T_2 (0, r)}{\partial x} \end{aligned} \right\} \tag{5}$$

Considering the solution of (2) in the region $x > 0$ and taking that T uniformly tends to some function T_∞ independent of x as $x \to \infty$, the equation (2) is simplified to

$$\frac{1}{r}\frac{d}{dr}\left(\frac{dT_\infty}{dr}\right)=f(r) \tag{6}$$

where

$$f(r)=\frac{-\mu}{k}\left(\frac{du}{dr}\right)^2$$

while writing (6) the term due to heat generation is neglected.

Thus the general solution of (6) using the proper boundary conditions (3) and (4) will be

$$T^+_\infty = T^+_a + P_{ra}(s) + \frac{ln\,(a/r)}{ln\,s}\left[\left((T^+_a - T^+_b) + P_{ab}(s)\right]\right. \text{ for } x>0 \tag{7}$$

and

$$T^-_\infty = T^-_a + P_{ra}(s) + \frac{ln\,(a/r)}{ln\,s}\left[(T^-_a - T^-_b) + P_{ab}(s)\right] \text{ for } x<0 \tag{8}$$

where

$$P_{\beta l}(s)=\int_0^\beta (\beta - s)\, f(s)\, ds - \int_0^l (l-s)\, f(s)\, ds \tag{9}$$

The solutions $T^{\pm}_\infty$ also particular solutions of (2) for the regions $x>0$ and $x<0$ respectively.

By introducing the following dimensionless variables in (2)

$$\frac{T-T_s}{T_0-T_s}=\theta,\ r=\eta\, a,\ x=Pe'\, a\xi \tag{10}$$

where Pe' is the Peclet number, (2) is reduced to

$$\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\theta}{\partial\eta}+\frac{1}{Pe'^2}\frac{\partial^2\theta}{\partial\xi^2}-\frac{2}{M}\left[(1-\eta^2)+N\,ln\eta\right]\frac{\partial\theta}{\partial\xi} = A + B\left(\frac{N}{\eta}-2\eta\right)^2 \tag{11}$$

where

$$\left.\begin{aligned} A&=\frac{q}{\rho ck\,(T_s-T_0)},\quad B=\frac{4\nu\, U^2_m}{kcM^2\,(T_0-T_s)}\\ M&=(s^2+1-N),\ N=\frac{s^2-1}{ln\, s}\end{aligned}\right\}. \tag{12}$$

The boundary conditions now become:

$$\theta=0 \text{ at } \eta=1, \text{ and } \eta=0 \text{ for } \xi>0, \tag{13}$$

$$\theta=1 \text{ in } 1<\eta<s \text{ for } \xi=0, \tag{14}$$

$$\theta\to\theta_\infty \text{ as } \xi\to\infty \text{ for } 0<\eta<1, \tag{15}$$

where

$$\theta_\infty=\frac{ln\,\eta}{4\,ln\,s}\left[A(1-s^2)+B\left\{2N^2\,(ln s)^2-4N\,(s^2+1)+s^4+1\right\}\right]+ + \frac{1}{4}\left[A\,(\eta^2-1)+B\left\{2N^2\,(ln\eta)^2-4N\,(\eta^2-1)+\eta^4-1\right\}\right]. \tag{16}$$

$$\text{Let}\quad T_1(\xi,\eta)=T^-_\infty(\eta)+\theta^*_1\,(\xi,\eta) \tag{17}$$

and $$T_2(\xi, \eta) = T^{+}_{\infty}(\eta) + \theta^{*}_{2}(\xi, \eta) \tag{18}$$

then θ^*_1 and θ^*_2 satisfy the equation (11) for the regions $\xi < 0$ and $\xi > 0$ respectively.

The solution of (11) can be written as

$$\theta = e^{-\lambda\xi}\,\psi(\eta) + \theta_{\infty} \tag{19}$$

where $\psi(\eta)$ satisfies the equation of the sturm-Liouville form

$$\frac{d^2\psi}{d\eta^2} + \frac{1}{\eta}\,\frac{d\psi}{d\eta} + \left[\frac{\lambda^2}{Pe^2} + \frac{2\lambda}{M}\,(1 - \eta^2 + N\,ln\eta)\right]\psi = 0 \tag{20}$$

Since the origin is exluded from the field the eigen function in this case will be

$$\psi(\eta) = \sum_{n=1}^{\infty} a_n \left[J_0(\alpha_n \eta)\,Y_0(\alpha_n) - J_0(\alpha_n)\,Y_0(\alpha_n \eta)\right] \tag{21}$$

where J_0 and Y_0 are Bessel functions of first and second kind.

It is clear that $\psi(\eta)$ vanishes when $\eta = 1$ and $\eta = s$, provided α_n is a root of

$$\left[J_0(\alpha_n s)\,Y_0(\alpha_n) - J_0(\alpha_n)\,Y_0(\alpha_n s)\right] = 0 \tag{22}$$

Equation (20) has two sets of complete eigen functions, one corresponding to the positive eigen values denoted by λ_n+, and the other to the negative set of eigen values λ_n^-.

In terms of the separate solutions, we have

$$\theta^*_1(\xi, \eta) = \sum_{\lambda^-_n} a^-_n\,\psi^-_n(\eta)\,e^{-\lambda^-_n\xi} \text{ for } \xi < 0 \tag{23}$$

where the a_n^- are constants and ψ_n^- is the eigen function corresponding to λ_n^-. The summation extends only over the λ_n^-.

Similarly

$$\theta^*_2(\xi, \eta) = \sum_{\lambda^+_n} a^+_n\,\psi^+_n(\eta)\,e^{-\lambda^+_n\xi} \tag{24}$$

which holds in $\xi > 0$.

For the sake of convenience, we shall write henceforth $a_n^{\pm} = a_n$ and $\psi_n^{\pm} = \psi_n$.

Multiply (20) by $\eta\,J_0(\alpha_m \eta)\,Y_0(\alpha_m)$ and integrate w.r.t. η between the limits 0 to 1, similarly multiply (20) by $\eta\,J_0(\alpha_m)\,Y_0(\alpha_m \eta)$ and integrate w.r.t. η between the limits 0 to 1 and substricting these expressions,* we get the following equation involving the coefficients a_n and eigen values λ'_s

$$\sum_{n=1}^{\infty} a_n \Bigg[\left(\frac{\lambda^2}{P_e^2} + \frac{2\lambda}{M} - \alpha_n^2\right)\frac{1}{2}\left\{s^2 L_1^2(\alpha_n s) - L_1^2(\alpha_n)\right\} +$$

$$+ \frac{\lambda}{M}\left\{\frac{J_0(\alpha_n)\,Y_0(\alpha_n)}{3\alpha_n^4}\right.\Bigg[- s^4 g_{12}(\alpha_n s) + g_{12}(\alpha_n) +$$

$$+ 3\left\{s^4 g_{10}(\alpha \quad) - g_{10}(\alpha_n)\right\}\Bigg] + \frac{2}{3\alpha^2_n}\Bigg[s^2 R_1(\alpha_n s) -$$

*In this process ∞_m need not be the sam igen value as ∞_n

$$- R_1(a_n) + a_n \left\{ - s^4 S_1(a_n s) + S_1(a_n) \right\} \Big] +$$

$$+ \frac{1}{3} \left[- s^4 R_1(a_n s) + R_1(a_n) - s^4 R_0(a_n s) + R_0(a_n) \right] +$$

$$+ N \Big[s^2 L_1^2(\alpha_n s) \ln s - s^2 R_1(\alpha_n s) + R_1(\alpha_n) +$$

$$+ \frac{1}{2} \left\{ s^2 S_2(a_n s) - S_2(a_n) \right\} + 2 \left\{ s^2 K_1(a_n s) - K_1(a_n) \right\} -$$

$$- \frac{1}{2} \left\{ s^2 g_{20}(\alpha_n s) - g_{20}(\alpha_n) \right\} J_0(\alpha_n) Y_0(\alpha_n) \Big]\Big\}\Big] +$$

$$+ \sum_{n=1}^{\infty} \frac{8\lambda}{M} \alpha_n \Big[- s^2 L_1(\alpha_n s) \sum_{m=1}^{\infty} \frac{\alpha_m L_1(\alpha_m s)}{(\alpha_n^2 - a_m^2)^2} a_m +$$

$$+ L_1(a_n) \sum_{m=1}^{\infty} \frac{\alpha_m L_1(\alpha_m)}{(a_n^2 - \alpha_m^2)^2} a_m -$$

$$- s L_1(\alpha_n s) \sum_{m=1}^{\infty} \frac{\alpha_m^2 L_2(\alpha_m s)}{(\alpha_n^2 - \alpha_m^2)^3} a_m +$$

$$+ L_1(a_n) \sum_{m=1}^{\infty} \frac{\alpha_m^2 L_2(\alpha_m)}{(a_n^2 - a_m^2)^3} a_m +$$

$$+ \alpha_n \Big\{ s L_2(a_n s) \sum_{m=1}^{\infty} \frac{a_m L_1(a_m s)}{(\alpha_n^2 - \alpha_m^2)^3} a_m -$$

$$- L_2(a_n) s \sum_{m=1}^{\infty} \frac{a_m L_1(a_m)}{(a_n^2 - a_m^2)^3} a_m \Big\}\Big] = 0. \quad (m \neq n) \qquad (25)$$

The otations used in (25) are given by

$$L_i(a_n s) = J_i(a_n s) Y_0(a_n) - Y_i(a_n s) J_0(a_n)$$

$$g_{ij}(a_n s) = J_i(a_n s) Y_i(a_n) + J_j(a_n s) Y_j(a_n)$$

$$R_i(a_n s) = J_i^2(a_n s) Y_0^2(a_n) + Y_i^2(a_n s) J_0^2(a_n)$$

$$S_i(\alpha_n s) = J_i(a_n s) J_0(a_n s) Y_0^2(a_n) + Y_i(a_n s) Y_0(a_n s) J_0^2(a_n)$$

$$K_1(\alpha_n s) = J_1(a_n s) Y_1(a_n s) J_0(a_n) Y_0(a_n),$$

where i, j take values 1, 2 respectively.

In obtaining (25), use of condition (13) has been made. The condition for the non-vanishing of a_n gives rise to the infinite determinant

$$\triangle(\lambda) = 0 \qquad (26)$$

The diagonal elements of this infinite determinant contain a quadratic expression in λ, and therefore (26) has an infinity of positive and negative roots, Let the relevant roots be $\lambda_p (p = 1,2\ldots\ldots)$ and corresponding to each λ_p there are infinite number of coefficients $a_n{}^p$ $(n = 1,2\ldots\ldots\ldots)$. [to be obtained from (25) after replacing λ by λ_p and a_n by $a_n{}^p$. All the constants $a^p{}_n$ can be expressed in terms of $a^p{}_p$.

It now remains to calculate $a^p{}_p$. Hence if we write

$$\theta = \sum_{m=1}^{\infty} X_m(\xi) \left[J_0(\alpha_m \eta)\, Y_0(\alpha_m) - J_0(\alpha_m)\, Y_0(\alpha_m \eta) \right] + \theta_\infty \tag{27}$$

where

$$X_m(\xi) = \sum_{p=1}^{\infty} a^p{}_m \, e^{-\lambda_p \xi} \tag{28}$$

and using the boundary condition (14), we have

$$1 = \sum_{m=1}^{\infty} \left\{ \left[J_0(\alpha_m \eta)\, Y_0(\alpha_m) - J_0(\alpha_m)\, Y_0(\alpha_m \eta) \right] \cdot \sum_{p=1}^{8} a^p{}_m \right\} + \theta_\infty \tag{29}$$

Multiply (29) by $\eta\, J_o(\alpha_m \eta)\, Y_o(\alpha_m)$ and integrate w.r.t. η between the limits 0 to 1, similarly multiply (29) by $\eta J_0(\alpha_m) Y_0(\alpha_m \eta)$ and integrate w.r.t. η between the limits 0 to 1 and subtracting them, it reduces to the following expression since $a^p{}_m$ can be expressed in terms of $a^m{}_m$

$$a^m{}_m F_m = \frac{sL_1(\alpha_m s) - L_1(\alpha_m)}{\alpha_m} \; \frac{A}{2\alpha_m{}^3} \left[2\, sL_1(\alpha_m s) - L_2(\alpha_m) \right] -$$
$$- \frac{B}{4\alpha_m} \left[2\, sL_1(\alpha_m s) \left\{ 2\, s^2 (s^2 - 1) - 2N (2\, s^2 - Nln\, s) + \right. \right.$$
$$\left. + 1 + \frac{8}{\alpha_m{}^2} (N - s^2) + \frac{4}{\alpha_m{}^2} \right\} +$$
$$\left. + \frac{4}{\alpha_m} L_2(\alpha_m)(1 - 2N) - \frac{8 L_3(\alpha_m)}{\alpha_m{}^2} \right] \tag{30}$$

where F_m is a known function of λ_m and α_m. These equations determine $a^m{}_m$, hence eigen functions and eigen values are thus obtained.

The complete solution of the problem is thus given by

$$T_1(\xi, \eta) = T_\infty^-(\eta) + \sum_{\lambda_n^-} a_n^- \psi_n^-(\eta)\, e^{-\lambda_n^- \xi}, \qquad \text{for } \xi < 0$$

and

$$T_2(\xi, \eta) = T_\infty^+(\eta) + \sum_{\lambda_n^+} a^+{}_n \, \psi^+{}_n(\eta)\, e^{-\lambda_n^+ \xi} \qquad \text{for } \xi < 0.$$

Graphical representation of the temperature profiles is provided in Fig. 1. Typical isotherms for $Pe=1$ show that there are marked variations near $\xi=0$, while they are almost constant elsewhere. On the other hand isotherms for $Pe=10$ almost parallel to ξ-axis. From the computations of profiles it is seen that they are parallel to ξ-axis for all other higher Peclet numbers. The Fig. 1 could be supplemented by Table 1 giving the Eigen values for a set of Peclet numbers.

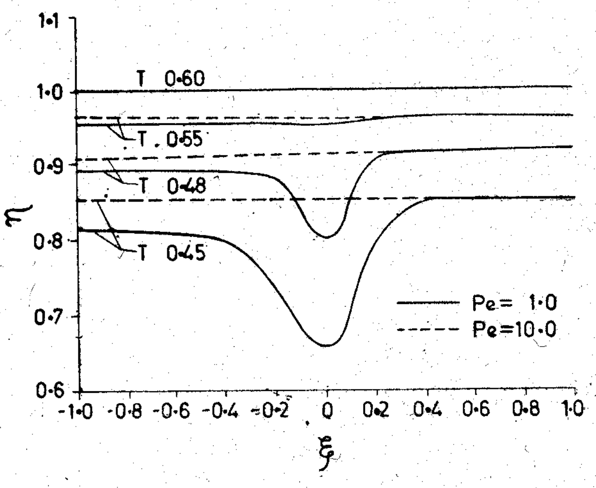

Fig. 1 Temperature profiles for $Pe = 1.0$ and $Pe = 10.0$.

TABLE 1

The Eigen valves for a set of peclet numbers.

Pe	λ_1	λ_2
1.0	7.64	— 5.96
5.0	60.88	—18.68
10.0	192.45	—23.65
50.0	4246.96	—26.87

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