LAMINAR SOURCE FLOW OF SECOND ORDER FLUID BETWEEN TWO PARALLEL COAXIAL DISKS ROTATING AT DIFFERENT SPEEDS

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The laminar source flow of second order fluid between two infinite parallel disks rotating with different velocities has been studied in this paper. The solution has been obtained by double series expansion about a known solution at a large radius. The effect of non-Newtonian parameters has been discussed on velocity profiles for small rotational Taylor numbers of the disks.

Laminar source flow between two parallel disks rotating at the same speed, has been investigated by Breitner & Pohlhausen[1] and Kreith & Peube[2, 3]. Pelech & Shapiro[4] have obtained an approximate first order solution when one disk is rotating and the other is stationery as a byproduct of an investigation of the deflection of a rotating magnetic recording device. Kreith & Viviand[5] considered axisymmetric flow between two disks, rotating at different angular velocities with a source at the centre. The Reynolds number Re based on source strength and the Taylor numbers α_i $(i=1,2)$ associated with angular velocity of each disk, are all assumed to be small. The problem is solved by expanding the flow variables in powers of Re about a known solution. This solution is also represented in a series expansion with α_i as perturbation parameter. The results are valid for small values of α_i and at a distance $r >> Re^{1/2}/\alpha_i$. Recently it has been shown by Mellor[6] et al., that the steady flow of a viscous incompressible fluid between two coaxial infinite disks, one rotating and the other at rest with zero source flow, does not give a unique solution.

The source flow of an incompressible second order fluid between two parallel disks, one rotating and the other at rest, has been investigated by Rajvanshi[7]. The constitutive equations of an incompressible second order fluid as suggested by Coleman & Noll[8] are :

$$\tau_{ij} = -p\, g_{ij} + \phi_1 A_{ij} + \phi_2 B_{ij} + \phi_3 A_{ik} A_{kj}, \tag{1}$$

$$A_{ij} = v_{i,j} + v_{j,i}, \tag{2}$$

and

$$B_{ij} = a_{i,j} + a_{j,i} + 2v_{m,i}\, v_{m,j}. \tag{3}$$

Where τ_{ij} is the stress tensor ; g_{ij}, the metric tensor ; v_i, the velocity vector ; a_i, the acceleration vector ; p, the pressure ; ϕ_1, ϕ_2, ϕ_3, the fluid parameters and comma(,) denotes covariant differentiation. Rajvanshi[7] obtained the solution by expanding flow variables in powers of the radius vector. The functions in the series expansion have been determined for small Reynold number depending on the angular velocity of rotation of the disk.

In the present paper the source flow of the second order fluid, between two parallel coaxial disks rotating at different speeds, defined by (1), (2), and (3), has been investigated. The series expansion method of Kreith & Viviand[5] has been adopted in this paper to obtain the solution. We note that the solution for zero source strength is not affected by the non-Newtonian parameters upto second order terms in Taylor number. But the effect of non-Newtonian parameters is present in the higher order approximations. In particular, it has been observed that the non-Newtonian parameters are most effective in perturbation terms of radial velocity when the two disks rotate in opposite direction, and the least, when the disks rotate in same direction with equal velocity.

EQUATIONS OF MOTION

The momentum equation for the incompressible flow is

$$\rho\, v_j\, v_{i,j} = \tau_{ij,j}\ , \tag{4}$$

and the equation of continuity is

$$v_{i,i} = 0. \tag{5}$$

We shall use polar cylindrical coordinates ($\bar{r}, \bar{\theta}, \bar{z}$) in the following discussion. The flow takes place between two parallel disks $z = +a$ and $z = -a$. Let the upper and lower disks rotate at angular velocities ω_2 and ω_1 respectively. The volumetric flow rate of the source (Strength) is to be assumed as Q.

Therefore the boundary conditions become

$$\left.\begin{aligned} &\bar{u} = \bar{w} = 0 \quad \text{at} \quad \bar{z} = \pm a, \\ &\bar{v} = \bar{r}\omega_1 \quad \text{at} \quad \bar{z} = -a, \\ &\bar{v} = \bar{r}\omega_2 \quad \text{at} \quad \bar{z} = +a; \\ &\int_{-a}^{+a} 2\pi \bar{r}\,\bar{u}\,d\bar{z} = Q. \end{aligned}\right\} \tag{6}$$

Let us introduce the following non-dimensional quantities

$$\left.\begin{aligned} &r = \bar{r}/a, \quad z = \bar{z}/a, \\ &u = \frac{\rho \bar{u} a}{\phi_1}, \quad v = \frac{\rho \bar{v} a}{\phi_1}, \quad w = \frac{\rho \bar{w} a}{\phi_1}, \\ &p = \frac{\rho \bar{p} a^2}{\phi_1^2}, \quad K = \frac{\phi_2}{\rho a^2}, \quad S = \frac{\phi_3}{\rho a^2}. \end{aligned}\right\} \tag{7}$$

Equations (6) and (7) now give the boundary conditions in the following form

$$\left.\begin{aligned} &u = w = 0 \quad \text{at } z = \pm 1, \\ &\int_{-1}^{+1} u\,dz = \frac{2\,Re}{r}, \\ &v = \alpha_1 r \quad \text{at } z = -1, \\ &v = \alpha_2 r \quad \text{at } z = +1. \end{aligned}\right\} \tag{8}$$

where

$$Re \text{ (Reynolds number)} = \frac{Q\rho}{4\pi a \phi_1},$$

$$\alpha_1 \text{ (Taylor number for lower disk)} = \frac{\omega_1 a^2 \rho}{\phi_1},$$

$$\alpha_2 \text{ (Taylor number for upper disk)} = \frac{\omega_2 a^2 \rho}{\phi_1}.$$

Let us define a stream function ψ as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{9}$$

Following Kreith & Viviand[5], the solution, which is valid for large values of $r/Re^{1/2}$, is assumed in the following form :

$$\psi = r^2 f_{-1}(z) + Re\left\{f_1(z) + \frac{Re}{r^2} f_3(z) + \ldots\right\}, \tag{10}$$

$$p = r^2 h_{-2}(z) + h_0(z) + Re\left\{h(z)\ln r + \frac{Re}{r^2} h_2(z) + \ldots\right\}, \tag{11}$$

and

$$v = r\, g_{-1}(z) + Re^{1/2} \left\{ \frac{Re^{1/2}}{r}\, g_1(z) + \frac{Re^{3/2}}{r^3}\, g_3(z) + \ldots \right\}. \tag{12}$$

Let us convert (1) to (5) into cylindrical polar coordinate system, and non-dimensionalise them by using (7). Substituting (9) to (12) in these non-dimensionalised equations and equating the coefficients of like powers of r, we obtain an infinite set of simultaneous ordinary differential equations. The first two systems are :

System I

$$f'''_{-1} + 2 f_{-1} f''_{-1} - f'^2_{-1} = 2\, h_{-2} - g^2_{-1} + K\,[\, f_{-1} f^{iv}_{-1} - 2 f''^2_{-1}\,] + S\,[\, g'^2_{-1} - 2 f''^2_{-1} + 2 f'_{-1} f'''_{-1}\,] \tag{13}$$

$$g''_{-1} + 2 f_{-1} g'_{-1} - 2\, g_{-1} f'_{-1} = K\,[\, 2 f_{-1} g'''_{-1} - 2 f''_{-1} g'_{-1}\,] + S\,[\, 2 f'_{-1} g''_{-1} - 2 f''_{-1} g'_{-1}\,] \tag{14}$$

$$h'_{-2} = 2\,(2K + S)\,[\, f'''_{-1} f''_{-1} + g''_{-1} g'_{-1}\,]. \tag{15}$$

The differential equation for h_0 is

$$h'_0 = -4 f'_{-1} f_{-1} - 2 f''_{-1} + 4K\,[\, f'''_{-1} f_{-1} + 11 f''_{-1} f'_{-1} + Re\,(f'''_{-1} f''_1 + g'_1 g''_{-1} + g''_1 g'_{-1} + f''_{-1} f'''_1)\,] + 2S\,[\, 14 f''_{-1} f'_{-1} + Re\,(g''_1 g'_{-1} + g'_1 g''_{-1} + f'''_{-1} f''_1 + f''_{-1} f'''_1)\,]. \tag{16}$$

System II

$$f'''_1 + 2 f_{-1} f''_1 = h - 2\, g_{-1} g_1 + 2K\,[\, f'_{-1} f'''_1 + f_{-1} f_1^{iv} + 2 g''_{-1} g_1 + 2 g'_1 g'_{-1} + f''_1 f''_{-1} + f'_1 f'''_{-1}\,] + 2S\,[\, f'_{1-1} f'''_{-1} + g_1 g''_{-1} + f''_{-1} f''_1 + f'_{-1} f'''_1 + 2 g'_{-1} g'_1\,], \tag{17}$$

$$g''_1 + 2 g'_1 f_{-1} = 2\, g_{-1} f'_1 + 2K\,[\, -2 f'_1 g''_{-1} + g'''_1 f_{-1} + g''_1 f'_{-1} + g'_1 f''_{-1} + g_1 f'''_{-1} - 2 f''_1 g'_{-1}\,] + 2S\,[\, g_1 f'''_{-1} - 2 f'_1 g'_{-1} - f'_1 g''_{-1} + f'_{-1} g''_1 + g'_1 f''_{-1}\,], \tag{18}$$

and

$$h' = 0. \tag{19}$$

The boundary conditions in the modified form are :

$$\left.\begin{aligned} f'_n(\pm 1) &= 0, \quad \text{for} \quad n = -1, 1, 3, \ldots, \\ f_n(\pm 1) &= 0, \quad \text{for} \quad n = -1, 3, \ldots, \end{aligned}\right\} \tag{20}$$

$$f_1(1) - f_1(-1) = 2. \tag{21}$$

We choose

$$f_1(-1) = 0,$$

so that

$$f_1(1) = 2. \tag{22}$$

$$\left.\begin{aligned} g_s(\pm 1) &= 0, \quad \text{for} \quad s = 1, 3, \ldots \\ g_{-1}(-1) &= \alpha_1 ; \text{ and} \\ g_{-1}(+1) &= \alpha_2 . \end{aligned}\right\} \tag{23}$$

In the above equations the prime denotes the differential coefficient with respect to z. The equation of continuity is identically satisfied by (9).

RESULTS & DISCUSSION

System I

We take Taylor series expansion for f_{-1}, g_{-1} and h_{-2} in the following form :

$$\left.\begin{aligned} f_{-1} &= \alpha_1^2 F_{11} + 2\,\alpha_1\,\alpha_2\,F_{12} + \alpha_2^2 F_{22} + \ldots \\ g_{-1} &= \alpha_1 G_1 + \alpha_2 G_2 + \alpha_1^2 G_{11} + 2\,\alpha_1\,\alpha_2\,G_{12} + \alpha_2^2 G_{22} + \ldots \\ h_{-2} &= \alpha_1^2 H_{11} + 2\,\alpha_1\,\alpha_2\,H_{12} + \alpha_2^2 H_{22} + \ldots \end{aligned}\right\} \qquad (24)$$

We substitute (24) in the equations of system I. On equating equal powers of α_1, α_2, and their products, we obtain four sub-systems of differential equations. These are readily solvable and the solutions are:

$$f_{-1} = -\frac{1}{240}\,(z^2 - 1)^2\,[\,\alpha_1^2\,(z-1) - 2\,\alpha_1\,\alpha_2\,z + \alpha_2^2\,(z+5)\,] + O(\alpha^4), \qquad (25)$$

$$g_{-1} = -\frac{1}{2}\,\alpha_1\,(z-1) + \frac{1}{2}\,\alpha_2\,(z+1) + O(\alpha^3), \qquad (26)$$

$$h_{-2} = \frac{3}{20}\,(\alpha_1^2 + \alpha_2^2) + \frac{1}{5}\,\alpha_1\,\alpha_2 + O(\alpha^4), \qquad (27)$$

where $O(\alpha^n)$ meaus terms of $O(\alpha_1^n)$ or (α_2^n). We have also from (25)

$$f'_{-1} = -\frac{1}{240}\,(z^2-1)\,[\,5\,(z^2-1)\,(\alpha_2-\alpha_1)^2 + 20\,(\alpha_2^2 - \alpha_1^2)\,z\,] + O(\alpha^4), \qquad (28)$$

which agrees with Kreith & Viviand[5]. Hence for zero source strength, the flow pattern remains unaffected by non-Newtonian fluid parameters upto second order terms in the Taylor number α_i $(i = 1, 2)$. It is sufficient to consider the value of α $(=\alpha_1/\alpha_2)$ between -1 and $+1$, since changing α into $1/\alpha$ gives the flow field upside down. These results are valid for small values of α_1 and α_2.

Equation (25) shows that w is always positive if $-2/3 < \alpha < 1$, but changes sign once if $-1 \leqslant \alpha < -2/3$, at a value of z equal to $z^* = 5\,\dfrac{\alpha+1}{\alpha-1}$. Kreith & Viviand[5], have shown that the plane $z = z^*$ is a stream surface which divides the flow field into two separate regions. The dividing stream surface, when it exists, rotate with the dimensionless angular velocity $\alpha^* = -2\,\alpha_2\,(\alpha+1)$ in the direction as the lower disk rotates. Since u is not zero at $z = z^*$, the dividing plane is not analogous to a solid disk. When α increases from -1 to $(-2/3)$, (α^*/α_1) increases from 0 to $+1$, whereas (α^*/α_2) decreases from 0 to $(-2/3)$. Casal[9] pointed out that for $\alpha_i = 0$, the series solution converges when $|\,\alpha_2\,| < 0{\cdot}17$.

System II

We assume the Taylor series expansion for f_1, g_1 and h in the form

$$\chi = \chi_0 + \alpha_1\,\chi_1 + \alpha_2\,\chi_2 + \alpha_1^2\,\chi_{11} + 2\,\alpha_1\,\alpha_2\,\chi_{12} + \alpha_2^2\,\chi_{22} + \ldots \qquad (29)$$

As the system possesses non-zero solution for $\alpha_1 = \alpha_2 = 0$, the zero order terms in (29) have been included.

By substituting (24) and (29) in system II [i.e. equations (17) to (19)], and equating like powers of Taylor numbers, we obtain a set of ordinary linear differential equations connecting the functions assumed in (29). These equations have been solved using modified boundary conditions. The solution in the final form is :

$$f_1 = \frac{1}{2}\,[\,2 + 3z - z^3\,] + \frac{1}{1120}\,(\alpha_1^2 + \alpha_2^2)\left[\,\frac{1}{9}\,z^9 - \frac{2}{5}\,z^7 - \frac{56}{5}\,z^5 + \frac{1042}{45}\,z^3 - \frac{35}{3}\,z\,\right] + \frac{1}{1120}\,(\alpha_1^2 - \alpha_2^2)\left[-\frac{1}{4}\,z^8 + 7\,z^6 - \frac{63}{2}\,z^4 + 43\,z^2 - \right.$$

$$- \frac{73}{4} \Big] + \frac{1}{280} \alpha_1 \alpha_2 \Big[- \frac{1}{18} z^9 + \frac{13}{15} z^7 - \frac{42}{5} z^5 + \frac{649}{45} z^3 - \frac{41}{6} z \Big] +$$

$$+ \frac{1}{80} (K + S) (\alpha_1^2 + \alpha_2^2) \Big[- \frac{1}{7} z^7 + \frac{23}{15} z^5 - \frac{277}{105} z^3 + \frac{131}{105} z \Big] +$$

$$+ \frac{1}{80} (K + S) (\alpha_1^2 - \alpha_2^2) [- z^6 - 5 z^4 + 13 z^2 - 7] + \frac{1}{4200} (K + S) .$$

$$. \alpha_1 \alpha_2 \Big[- 24 z^7 - \frac{49}{2} z^5 + 121 z^3 - \frac{145}{2} z \Big] + \frac{1}{10} (K + S)^2 (\alpha_1 - \alpha_2)^2 .$$

$$. [z^5 - 2 z^3 + z] , \tag{30}$$

$$g_1 = \frac{1}{40} (1 - z^2) [5 (\alpha_1 + \alpha_2) (z^2 - 5) + z (\alpha_2 - \alpha_1) (3 z^2 - 7)] +$$

$$+ (K + S) (\alpha_1 - \alpha_2) (z - z^3) \tag{31}$$

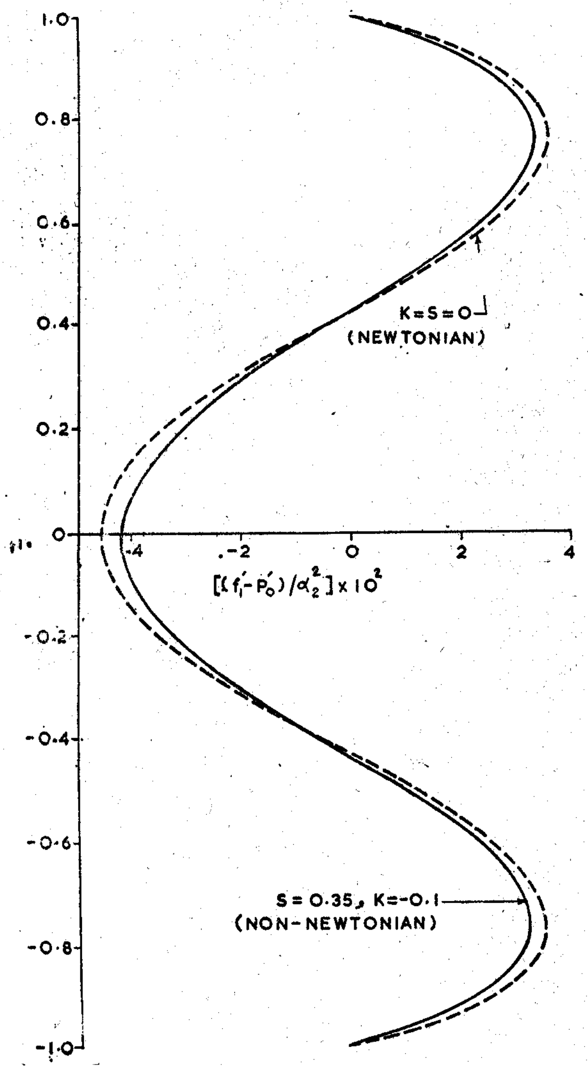

Fig. 1—Variation of $[(f_1'-Po')/\alpha_2^2]\times 10^2$ with z for $\alpha=1$.

and

$$h = - 3 - \frac{1}{525} [263 (\alpha_1^2 + \alpha_2^2) + 494 \alpha_1 \alpha_2 - 3 .$$

$$. (K + S) \{ 4 (\alpha_1^2 + \alpha_2^2) + 31 \alpha_1 \alpha_2 \}] + \frac{4}{5} .$$

$$. (K + S) (\alpha_1 - \alpha_2)^2 . \tag{32}$$

If we put $K = S = 0$ in (30) to (32), these agree with the corresponding equations of Kreith & Viviand[5]. The effect of non-Newtonian parameters has been shown graphically on the perturbation term of order α_2^2 in f_1'. In Fig. 1, $[(f_1' - P'_0)/\alpha_2^2] \times 10^2$ has been plotted against z for $\alpha = 1$ i.e. both the disks rotate in the same direction with same angular velocity. The curves have been drawn for $K = S = 0$ (Newtonian case) and $K = -0 \cdot 1$, $S = 0 \cdot 35$. We note that the profiles are symmetrical about $z = 0$. The effect of non-Newtonian parameter on $[(f'_1 - p'_0)/\alpha^2_2] \times 10^2$ is very small. Fig. 2 gives the profiles for (i) $K = S = 0$, (ii) $K = -0 \cdot 1$, $S = 0 \cdot 25$ and (iii) $K = -0 \cdot 1$, $S = 0 \cdot 35$ at $\alpha=0$, i.e. when the lower disk is stationary and upper sotates with given angular velocity. Inthis case, we note that the profile is not symmetrical about $z = 0$. The effect of non-Newtonian parameters is more marked in comparison to the case $\alpha = 1$. In Fig. 3, the profiles have been drawn for (i) $K = S = 0$, (ii) $K = -0 \cdot 1$, $S = 0 \cdot 25$, and (iii) $K = -0 \cdot 1$, $S = 0 \cdot 35$ at $\alpha = -1$, i.e. the two disks rotate in opposite directions with equal angular velocities. The profiles are symmetrical about $z = 0$. The effect of non-Newtonian parameters is more marked in comparison to $\alpha = 0$ and 1. We infer that the effect of non-Newtonian parameters is most conspicuous when torsion of the fluid takes place.

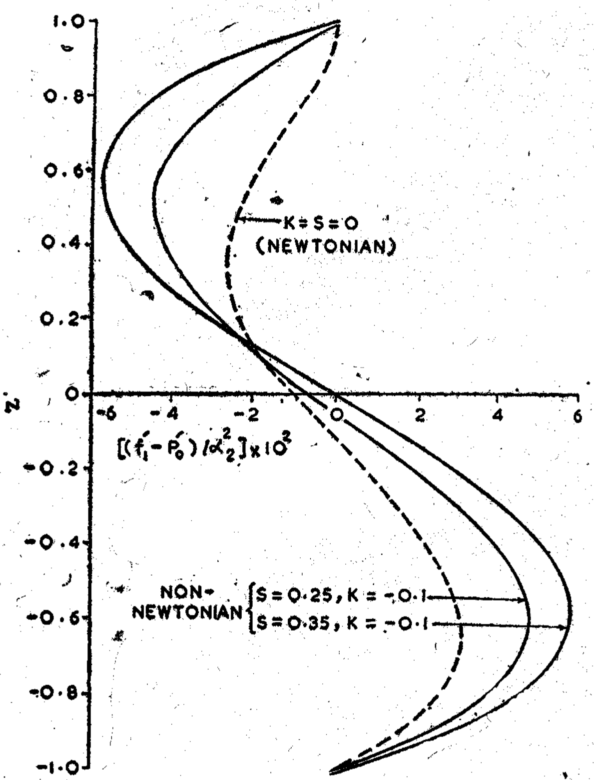

Fig. 2—Variation of [$(f_1'—Po')/a_2^2$] $\times 10^2$ with z for $a=0$.

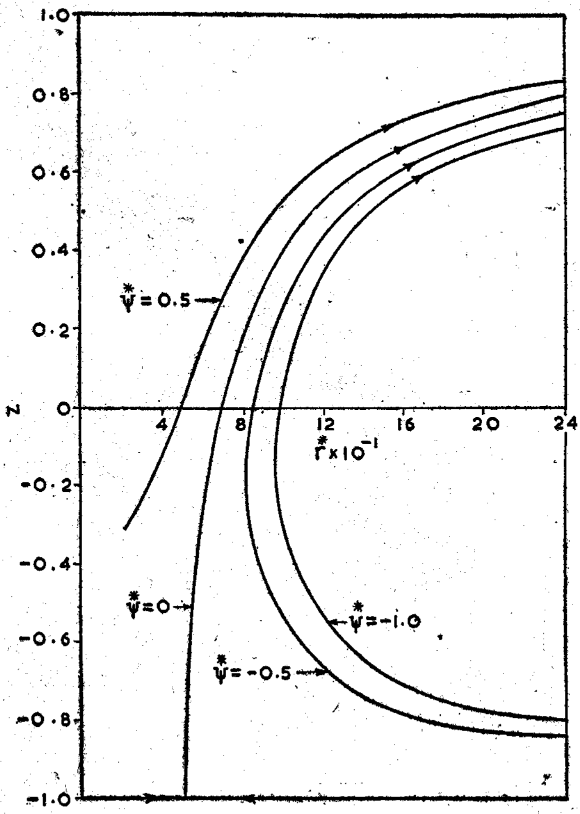

Fig. 4—Stream lines for $a=0$, $K = -0 \cdot 1$ & S=0 3.

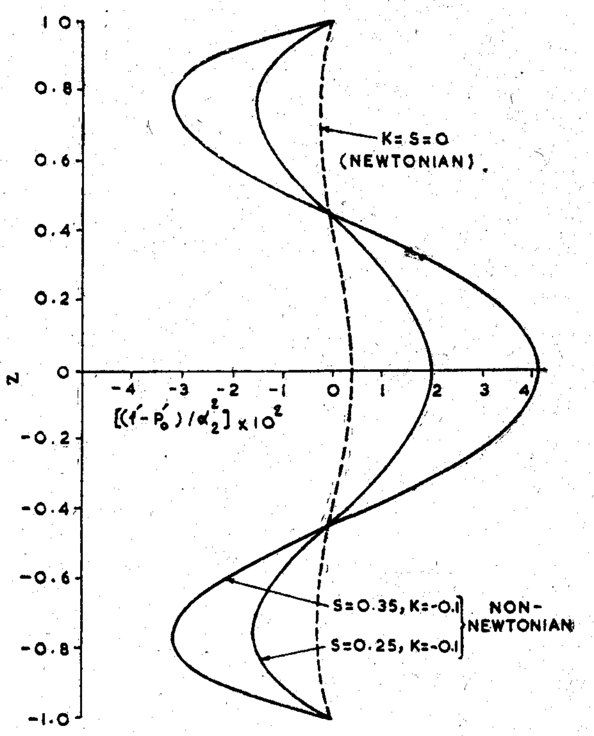

Fig. 3—Variation of [$(f_1'—Po')/a_2^2$) $\times 10^2$ with z for $a=—1$.

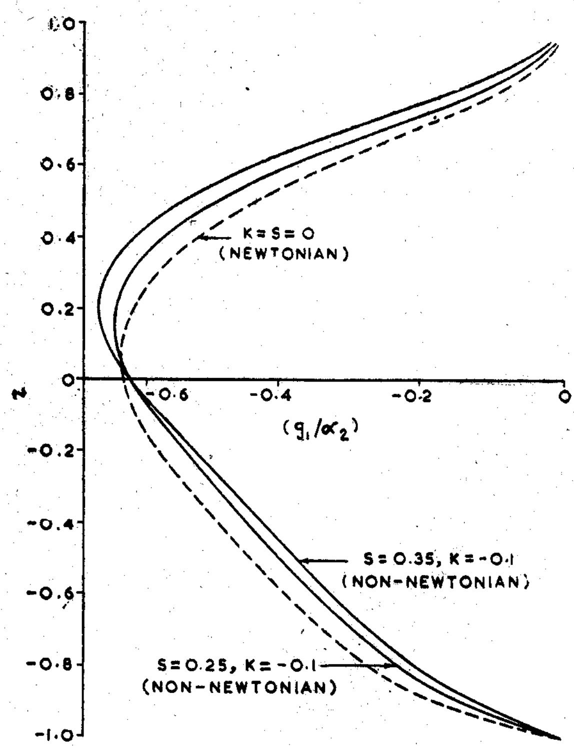

Fig. 5—Variation of (g_1/a_2) with z for $a=0$.

STREAMLINES

The stream function ψ is given by (10). This equation is rewritten in the following form :

$$\psi^* = r^{*2} f_{-1} + f_1 + \ldots \quad (33)$$

where $\psi^* = \psi \,/\, Re$ and $r^* = r \,/\, Re^{1/2}$.

The functions f_{-1} and f_1 are given by (25) and (30).

The streamlines for $K = -0{\cdot}1$ and $S = 0{\cdot}3$ have been shown in Fig. 4 for $\alpha_1 = 0$ and $\alpha_2 = -0{\cdot}2$. It is of the same pattern as for Newtonian case. If we neglect the rotation perturbation term in f_1', the stream function ψ is then simply the sum of its value for case of zero source flow and for the case of fixed disks. In this case the effect of non-Newtonian fluid parameters upto α_i^2 ($i = 1, 2$) is absent in the stream function. Therefore the streamlines are of the same form as drawn by Kreith & Viviand[5].

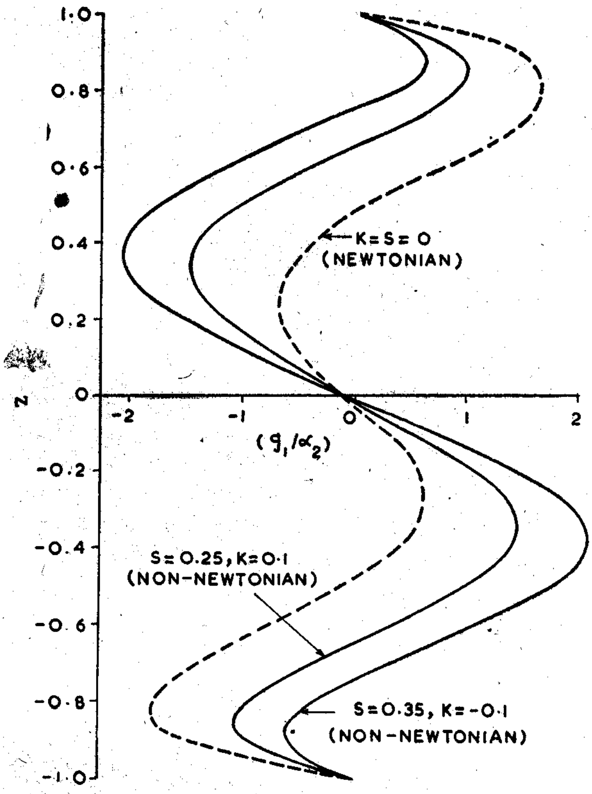

Fig. 6—Variation of (g_1/α_2) with z for $\alpha = -1$.

ANGULAR VELOCITY

Equations (12), (26) and (31) give the angular velocity

$$\frac{v}{r\,\alpha_2} = \frac{1-\alpha}{2}\, z + \frac{1+\alpha}{2} + \frac{Re}{r^2}\,(g_1 / \alpha_2), \quad (34)$$

where g_1 is given by (31).

The angular velocity profile will differ very little from a linear profile and the difference will vanish as r increases, since α_2 is assumed $< \alpha$. In Fig. 5 and 6, the effect of non-Newtonian parameters on (g_1 / α_2) for $\alpha = 0$ and $\alpha = -1$, has been exhibited. Here also we note that the effect is more at $\alpha = -1$ in comparison with that at $\alpha = 0$. It has been observed that the contribution of K and S to (g_1 / α_2) is negligible at $\alpha = 1$.

PRESSURE DISTRIBUTION

The pressure can be obtained from (11) by neglecting the term of order Re / r^2 or smaller. Then pressure upto second order terms in α_i ($i = 1, 2$) is given by

$$p = \frac{r^2\,\alpha_2^2}{20}\,[\,3\,(\alpha^2+1) + 4\,\alpha\,] - Re\, ln\, r\,\Big[\Big\{\,3 + \alpha^2_2\,\frac{263\,(\alpha^2+1) + 594\alpha}{525}\Big\} +$$

$$+ \frac{1}{4200}\,\alpha_2^2\,(\,K + S\,)\,(\alpha^2 + 1\,) + 744\,\alpha - \frac{4}{5}\,\alpha^2_2\,(\,\alpha - 1\,)^2\,(\,K + S\,)^2\,\Big] +$$

$$+ \frac{1}{120}\,\alpha_2^2\,(\,z^2 - 1\,)\,\big\{\,(\,5\,z^2 - 1\,)\,(\,1 - \alpha\,)^2 - 20\,(\,1 - \alpha^2\,)\,z\,\big\} +$$

$$+ \frac{1}{20}\,Re\,(\,2\,K + S\,)\,\Big[- z\,\alpha_2^2\,(\,1 - \alpha\,)\,\big\{\,5\,(\,z^2 - 5\,)\,(\,\alpha + 1\,) +$$

$$+ z(1-\alpha)(3z^2-7)\Big\} + (1-z^2)\alpha_2^2(1-\alpha)\Big\{5z(\alpha+1) +$$

$$+ \tfrac{1}{2}(1-\alpha)(3z^2-7) + 3z^2(1-\alpha)\Big\} - 20(K+S)(\alpha-1)^2(1-3z^2)\alpha_2 +$$

$$+ \frac{1}{20}\alpha_2^2(1-\alpha)z\Big\{(1-\alpha)(20z^3-12z) + (60z-20)\Big\}\Big] + \text{const.} \qquad (35)$$

The radial pressure distribution and the strength of the source are given by the first two terms as a function of the Taylor numbers for the upper and lower disks. The axial variation in pressure is given by the third term as a function of the axial distance from the centre plane.

The average normal force on the circular portion of the radius r_1 of the disk $z = 1$, is

$$\frac{1}{\pi r_1^2}\int_0^{r_1} 2\pi r(\tau_{zz})_{z=1}\,dr = \frac{\phi_1}{2a^2\rho}\Big\{r_1^2\Big[-\alpha_2^2\Big\{\frac{3}{20}(\alpha^2+1) + \frac{1}{5}\alpha - \frac{5}{8}(\alpha-1)^2\Big\} +$$

$$+ \tfrac{1}{4}\alpha_2^2(1-\alpha)^2\Big] - Re(2\ln r_1 - 1) - \Big[3 - \frac{1}{525}\alpha_2^2\,.$$

$$.\Big\{263(\alpha^2+1) + 494\alpha\Big\} - \frac{1}{175}(K+S)\alpha_2^2\Big\{4(\alpha^2+1) +$$

$$+ 31\alpha\Big\} + \frac{4}{5}\alpha_2^2(\alpha-1)^2(K+S)^2\Big]\Big\} + Q_0 \qquad (36)$$

where Q_0 is constant.

Hence the disk $z = 1$ experiences suction or thrust according as

$$r_1^2\alpha_2^2[9\alpha^2 - 38\alpha + 9] - 8Re(2\ln r_1 - 1)\Big[-15 - \frac{1}{105}\alpha_2^2\,.$$

$$.\{263(\alpha^2+1) + 494\alpha\} - \frac{1}{35}(K+S)\alpha_2^2\{4(\alpha^2+1) + 31\alpha\} +$$

$$+ 4\alpha_2^2(\alpha-1)^2(K+S)^2\Big] \lessgtr \frac{80Q_0a^2\rho}{\phi_1}. \qquad (37)$$

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