A BAYESIAN APPROACH TO A RENEWAL FUNCTION

D. D. BHATTACHARYA

Directorate of Scientific Evaluation

R. & D Organisation, Ministry of Defence, New Delhi-110001

(*Received* 21 *June* 1978)

The historical data on the consumption of spares are usually available in the form that the renewal function of the item could not easily be obtained by the process described by Cox[1]. In this paper, a technique is discussed as to how such data be used for the renewal functions through Bayesian approach. A Beta *a Priori* density for the parameter of Poisson demand is assumed and the Predictive demand distribution for such items is obtained. The renewal function in its analytical and asymptotic forms are also obtained.

The renewal function $H(t)$ as defined by Cox[1] is obtained from the fundamental relationship between the number of renewals N_t in fixed time (o, t) and the cumulative distribution $K_r(t)$ of the r^{th} renewal time of an item.

The limitation to this procedure is wellknown that whenever the integral transform of the failure time distribution of the item, is a rational function of the dummy variable of the transformation; the integral transform of the probability generating function (p.g.f.) of N_t can then be expanded in partial functions and hence inverted interms of elementary functions as indicated by Cox[1]. The analytical form of the distribution for N_t is thus not easily found and the renewal function in its asymptotic form is generally obtained.

On the otherhand, however, the consumption data and activity are usually available in the form as the number of demands for spare parts over some fixed period of operational time. For example, the data on consumption of aircraft spare parts are usually related to the aircraft months and that of vehicle to the vehicle months. Such historical data on consumption of spares, makes all the more difficult to obtain the renewal function on the lines as given by Cox[1]. Under such condition, firstly the past consumption data and activity are analysed and a suitable demand pattern is formulated. For example, Youngs *et al*[2] on examination of consumption of aircraft spare parts, observed that the demand for such items is characteristically low and the ability to predict in which of many weeks of operation, a particular item would be demanded is quite limited. Therefore, it has been suggested there that the Poisson distribution could be used to account for the variability in demand. However, for low demand rates, the estimate of true mean demand could not accurately be made from the observed data. Therefore, the approach outlined by Youngs *et al*[2] is that the demand for a given item follows the Poisson distribution, but that its Poisson Parameter value (mean demand) is unknown. Thus, an observed demand then becomes an outcome of the followings :

(*a*) First a sampling among the distribution of the Poisson Parameters and

(*b*) Second a sampling from the Poisson distribution with that parameter.

A so-called *a Priori* density of the Poisson Parameter is then Postulated on the basis that a large fraction of item observed to be concentrated at very low mean demand values for aircraft spare parts. Youngs *et al*[2], therefore, assumed a Gamma Prior for the Poisson parameter and obtained the conditional probability distribution that the demand for an item will be Y in a stipulated future period given that the same item has had X demands in a given past period.

Since, the mean demand values for aircraft spare parts are very low, in this present paper, instead of Gamma, a Beta—Prior density for the Poisson Parameter has been assumed. The successive past demands for such spare parts are then used to arrive at a revised probability density for the Poisson Parameter of the Poisson demand distribution on the Bayesian approach. The renewal function for the predictive demand distribution in the future period for the same item has then been obtained.

BAYESIAN APPROCH

The uncertainty involved for the evaluation of the true mean demand value in the Poisson demand distribution, is partly determined by one's own experience and partly by the data and evidence which one can obtain. When uncertainty is measured interms of Probability as a Prior density, then the Bayes theorem[3] provides a logic for computing the revised probability known as Posterior density for the uncertainty based on both one's experience and data. In fact, such guide line modifies one's own experience in a logical way and helps in reducing the uncertainty. Keeping this logic in view, the Bayesian approach has been used to formulate the renewal function for the interval $(r, r+1)$, i.e. the expected predictive demand in the $(r+1)$th time period given that the demands for the same item in r successive time periods each of equal length have been observed in the past.

DEMAND DISTRIBUTION

Under the assumption (*a*), let the Prior density for the mean demand $(1 > \lambda > 0)$ be governed by

$$f(\lambda) = \frac{\lambda^{\nu_1 - 1} (1 - \lambda)^{\nu_2 - 1}}{B(\nu_1, \nu_2)} \tag{1}$$

Where $\nu_1, \nu_2 > 0$ and $B(\,.\,,\,.\,)$ is the complete Beta function. The estimates of ν_1 and ν_2 are given in the Appendix.

Now, combining with the assumption (*b*), one obtains the Probability $P(X)$ that the demand for an item drawn from the population of items characterized by $f(\lambda)$ will be X in any fixed period of unit length (Such as aircraft months, Vehicle months) of experience, as follows :—

$$P(X = x) = \int_0^1 \frac{e^{-\lambda} \,.\, \lambda^x}{x\,!} f(\lambda)\, d\lambda ,$$

where $X = 0, 1, \ldots\ldots\ldots$

substituting from (1) for $f(\lambda)$, one gets

$$P(X = x) = \frac{B(\nu_1 + x, \nu_2)}{B(\nu_1, \nu_2)\, x\,!} \; {}_1F_1 (x + \nu_1 ; x + \nu_1 + \nu_2 ; -1) \tag{2}$$

Where, ${}_1F_1 (\,.\,;\,.\,; -1)$ is the confluent hypergeometric function[4].

It is further assumed that the demands for a particular item in the successive periods of operation are independent and identically distributed and that the mean demand for the item remains the same in all periods. Now, if the demands $x_1, x_2, \ldots\ldots\ldots\ldots, x_r$ for a particular spare part are observed in r successive periods each of unit length, the revised value for $f(\lambda)$ is then obtained by the Baye's theorem[5] as the posterior density for λ as

$$f^* \ (\lambda/X = x_1, x_2, \ldots, x_r) = \frac{e^{-r\lambda} \cdot \lambda^{S_r} \cdot f(\lambda)}{\int_0^1 e^{-r\lambda} \cdot \lambda^{S_r} \cdot f(\lambda)\, d\lambda} \quad , \quad S_r = \sum_{i=1}^{r} x_i$$

$$= \frac{e^{-r\lambda} \cdot \lambda^{S_r + \nu_1 - 1} \cdot (1-\lambda)^{\nu_2 - 1}}{B(s_r + \nu_1, \nu_2)\, {}_1F_1(S_r + \nu_1 ; S_r + \nu_1 + \nu_2 ; -r)} \tag{3}$$

Thus, the predictive demand distribution of an item in the period (r, $r+1$) i.e. for the $(r+1)^{th}$ class of unit operational period, given that the demands $x_1, x_2, \ldots\ldots, x_r$ units, for the same item have been observed in the past in r successive experience periods each of unit length, is given as

$$P(Y = y/X = x_1, x_2, \ldots, x_r) = \int_0^1 \frac{e^{-\lambda} \cdot \lambda^y}{y\,!} f^*(\lambda/X = x_1, x_2, \ldots, x_r)\, \alpha\lambda$$

$$= \frac{B(S_r + \nu_1 + y, \nu_2)\, {}_1F_1(S_r + \nu_1 + y ; S_r + \nu_1 + \nu_2 + y ; -r-1)}{B(S_r + \nu_1, \nu_2) \cdot y\,!\; {}_1F_1(S_r + \nu_1 ; S_r + \nu_1 + \nu_2 ; -r)} \tag{4}$$

THE RENEWAL FUNCTION

From equation (4), the p.g.f. of the Predictive demand distribution is obtained as

$$G(Z/X = x_1, x_2, \ldots, x_r) = \sum_{y=0}^{\infty} \frac{(S_r + \nu_1)_y\, Z^y\, {}_1F_1(S_r + \nu_1 + y ; S_r + \nu_1 + \nu_2 + y ; -r-1)}{(S_r + \nu_1 + \nu_2)_y \cdot y^1\, {}_1F_1(S_r + \nu_1 ; S_r + \nu_1 + \nu_2 ; -r)}$$

where

$$(a)_b = a.\,(a+1)\ldots\ldots(a+b-1).$$

Using multiplication theorem of ${}_1F_1$ (. ; . ;), the p.f.g. reduces to

$$G(Z/X = x_1, x_2, \ldots, x_r) = \frac{{}_1F_1(S_r + \nu_1 ; S_r + \nu_1 + \nu_2 ; -r-1+z)}{{}_1F_1(S_r + \nu_1 ; S_r + \nu_1 + \nu_2 ; -r)} \tag{5}$$

Now, on differentiation of (5) w.r.t.z. and letting $Z \to 1$, one finds the renewal function for the period (r, $r+1$) as

$H_{r,\, r+1}$ (t = unit period of operation)

$$= \frac{(S_r + \nu_1)\, {}_1F_1(S_r + \nu_1 + 1 ; S_r + \nu_1 + \nu_2 + 1 ; -r)}{(S_r + \nu_1 + \nu_2)\, {}_1F_1(S_r + \nu_1 ; S_r + \nu_1 + \nu_2 ; -r)} \cdot \tag{6}$$

ASYMPTOTIC VALUE OF THE RENEWAL FUNCTION

Let $\hat{\lambda}$ be the true mean demand of the Poisson demand Process of an item. Then for large number of successive intervals of operation r, one gets by using the weak Law of large number[6] that

$$\lim_{r \uparrow \infty} \left(\frac{S_r}{r}\right) \simeq \hat{\lambda} \quad \text{with the probability one.}$$

Where S_r is the total demand for particular item in r successive periods. Now, for large r, the asymptotic value of equation (6) is obtained by using[4] that

$${}_1F_1\,(a;\, b;\, -r) \simeq r^{-a}\ \frac{\lceil b}{\lceil b-a} \cdot \left\{1 + 0\ [1\, r\, 1^{-1}\,]\right\}$$

gives

$$\lim_{r \uparrow \infty} G\,(Z/X = x_1, x_2, \ldots\ldots, x_r) = {}^{-(1-z)} \cdot \frac{S_r}{r} = e^{-(1-z)\hat{\lambda}} \qquad (7)$$

and this is the p.g.f. of the Poisson demand distribution with the parameter $\hat{\lambda}$.

This proves that as the number of periods of experience increases, the renewal function for an item per unit of operation, becomes close to the true mean demand of the Poisson demand distribution.

EXAMPLE

Let the data on the consumption of aircraft spare parts are available for six periods each of T aircraft months and $\nu_1 = 0{\cdot}5$ and $\nu_2 = 0{\cdot}2$ (say) be obtained for airframe parts on the lines as given in appendix. It is now desired to find the renewal function of such item for the 7th period of operation when the consumption data for the same item is observed to be zero during the past six periods of experience.

Now, from equation (6), one gets.

$$H_{6,7}\ (T\text{=Unit period of Operation}) = \frac{5\ {}_1F_1\,(1{\cdot}5\,;\ 1{\cdot}7\,;\,-6)}{7\ {}_1F_1\,(0{\cdot}5\,;\,0{\cdot}7;\,-6)}$$

Using the recurrence formula[4] for ${}_1F_1$ (. ; . ; + 6), the above equation reduces to

$H_{6,7}$ (T=Unit period of Operation) = $0{\cdot}11$

Likewise, if the past consumption is *one* over the six periods of operation, the renewal function for such item for the 7th period is given as

$H_{6,7}$ (T=Unit period of Operation) = $0{\cdot}36$

CONCLUSION

In the military depot, it is the usual practice to stock the spare parts of a system for a fixed operational period; say, the period may be T_1 aircraft months and T_2 vehicle months for the spare parts of aircraft and vehicle respectively. In such cases, the present method will provide the necessary renewal function of an item of the system for its proper operation of fixed duration.

The value $H_{6,7}^{(T)} = 0{\cdot}11$ as shown in the example, indicates that even though there was no demand for the item during the past six successive operational periods, still there exists a chance of its being demanded in the next future period of planned operation. The value $H_{6,7}^{(T)} = 0{\cdot}36$ for the item shows its expected demand for the 7th period of operation of the system, given that the item has had one demand in the past six operational periods each of equal length.

Finally, the Beta *a Priori* density for the parameter of the poisson demand distribution is suitable for those spares where the mean demand for a fixed operational period is very small and less than unity.

ACKNOWLEDGEMENT

The author is grateful to Dr. S. S. Srivastava, Consultant, Directorate of Scientific Evaluation, R & D Organisation for suggesting the problem. Thanks are also due to the Director, Scientific Evaluation for the permission to publish the paper.

REFERENCES

1. COX, D.R., 'Renewal Theory' (Methuen & Co. Ltd. London), 1967 p. 36.
2. YOUNGS, J. W. T., GEISTER, M. A. & BROWN, B.B., The Prediction of demand for aircraft spare parts using the method of conditional probabilities. RAND Corpn., RM-1413, 17 Jan 1955.
3. MORRIS, WILLIAM, T., 'Management Science, a Bayesian Introduction' (Prentice-Hall Int. Series Management, London), 1968, p. 35.
4. SLATER, L. J., 'Confluent hypergeometric functions' (Cambridge University Press, London), 1960.
5. MARTIN, J. J., 'Bayesian Decision Problems and Markow Chains' (John Wiley & Sons Inc. New York), 1967, p. 16.
6. HEATHCOTE, C. R., 'Probability : Elements of the Mathematical theory' (Unwin University, Book., London), 1971, p. 169.

Appendix

ESTIMATION OF ν_1 and ν_2

The equation (2) gives the demand distribution of any item of the population of items whose mean demand is governed by $f(\lambda)$. Therefore, the items are first classified according to the sub-system; viz. airframe parts in case of aircraft, mechanical component of the vehicular engine etc.

Now let $N(x)$ be the actual number of items of the population with the observed demand x within a period of length equal to that for which the renewal function is obtained, and let $N = \sum_{x=0}^{\infty} N(x)$ be the total number of items. Therefore, the relative frequency of demand with which the demand is actually observed will relate to $P(x)$ given by equation (2).

Let

$$\left.\begin{aligned} \mu'_{[1]} &= \sum_{x=0}^{\infty} \frac{x\,N(x)}{N} \\ \mu'_{[2]} &= \sum_{x=0}^{\infty} \frac{x(x-1)\,N(x)}{N} \end{aligned}\right\} \qquad (i)$$

are the first two factorial moments of $P(x)$. The p.g.f. of the demand distribution $P(x)$ is given as

$$G(Z) = \sum_{x=0}^{\infty} Z^x\, P(x) = \frac{{}_1F_1\,(\nu_1\,;\,\nu_1+\nu_2\,;\,-1+Z)}{{}_1F_1\,(\nu_1\,;\,\nu_1+\nu_2\,;\,0)} \qquad (ii)$$

The solutions of (*i*) & (*ii*) give the estimates of ν_1 and ν_2 through $G'(Z)$ & $G''(Z)$ at $Z \to 1$ as

$$G'(1) = \frac{\nu_1}{\nu_1+\nu_2} \quad \text{and}$$

$$G''(1) = \frac{\nu_1(\nu_1+1)}{(\nu_1+\nu_2)(\nu_1+\nu_2+1)} \,.$$