# UNSTEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN PARALLEL PLATES

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In the present paper an exact solution of the Navier Stokes equations has been obtained, considering the flow of a viscous incompressible fluid between two infinitely extended parallel plates when upper plate is moving with uniform velocity and the lower plate is performing linear oscillations in its own plane. The technique of Laplace transform has been employed to obtain the velocity distribution, which has been shown graphically.

The steady motion of a viscous incompressible fluid between two infinitely extended parallel plates when upper one is moving with uniform velocity and the lower one is at rest, is a well known plane Couette flow[1]. The exact solutions of Navier Stokes equations for the flow near a plate with impulsive and simple harmonic motion are also well known[2]. The unsteady motion of a viscous incompressible fluid due to the periodic pressure gradient in different geometries has been considered by several researchers—Sexel[3], Uchida[4], Verma[5] and Drake[6]. Dube[7] has further investigated the unsteady flow of a viscous incompressible fluid in a channel bounded by the parallel flat plates. He has considered two cases: (i) When pressure gradient is varying linearly with time, (ii) When pressure gradient is decreasing exponentially with time.

Gupta & Goyal[8] have analysed the same problem for the first case of Dube[7]. They have reported a back flow initially for ( $3b_2 < b_1$ ). Recently Verma & Gaur[9] have investigated the unsteady flow and temperature distribution of a viscous incompressible fluid between parallel plates in which they have improved the result of Gupta & Goyal[8].

Studies of the unsteady problems involved with helicopter rotors (such as oscillating cross flow over the rotors, the time dependent oscillating flow over the airofoils etc.) are of much interest for minimising the unsteady effects of the environment. As it is well known, that the flow over a flat plate is a drastic simplification of the flow over a helicopter wing. To have an insight in such problems, we have considered the unsteady flow of a viscous incompressible fluid in a straight channel bounded by two infinitely extended parallel plates when the upper plate is moving with uniform velocity $U$, and the lower one is performing oscillations of the type $u = V e^{int}$ in its own plane. The method of Laplace transform is used to obtain the velocity distribution. The velocity profiles have been plotted in different cases to show the effect of oscillations. Coefficient of skin-friction is also plotted.

## FORMULATION OF THE PROBLEM AND ITS SOLUTION

We assume the fluid to be confined between two infinitely extended parallel plates at $y=0$ and $y=y_0$. Axis of $x$ be taken along the lower plate and $y$ be measured normal to it. The upper plate is moving in $x$ direction with uniform velocity $U$, and the lower plate is performing oscillations of the type $u=V e^{int}$.

The equations of motion and continuity for an incompressible viscous fluid in two-dimensions are :

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{1}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{2}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{3}$$

In the present problem $v=0$, therefore from (3), we have

$$\frac{\partial u}{\partial x} = 0$$

Also

$$\frac{\partial p}{\partial x} = 0 = \frac{\partial p}{\partial y}$$

Thus from equation (1), we have

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{4}$$

We shall solve the equation (4) by the technique of Lighthill[10] i.e. we assume the expression for velocity as

$$u = u_0 (y) + \epsilon \; u_1 (y, t) , \tag{5}$$

where $\epsilon$ is the perturbed parameter.

Substituting (5) in (4), we have

$$\epsilon \frac{\partial u_1}{\partial t} = \nu \left( \frac{\partial^2 u_0}{\partial y^2} + \epsilon \frac{\partial^2 u_1}{\partial y^2} \right), \tag{6}$$

On equating the coefficient of $\epsilon$ on both the sides in (6), we have the following set of equations :

$$\frac{\partial^2 u_0}{\partial y^2} = 0 , \tag{7}$$

and

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2} \tag{8}$$

Equation (7) is to be solved subject to the following boundary conditions :

$$\left. \begin{array}{l} u_0 = U ; \text{ at } y = y_0 , \\ u_0 = 0 ; \text{ at } y = 0 . \end{array} \right\} \tag{9}$$

The solution of (7) is

$$u_0 = \left( \frac{U}{y_0} \right) y \tag{10}$$

Now in order to solve (8), we define the Laplace transform

$$\bar{u}_1 (y, p) = \int_0^\infty e^{-pt} u_1 (y, t) \, dt \tag{11}$$

Equation (8) is to be solved subject to the following boundary conditions :

$$\left. \begin{array}{ll} u_1 = V e^{int} ; & \text{at } y = 0 \;\; ; t > 0 , \\ u_1 = 0 & ; \text{at } y = y_0 ; t > 0 , \\ u_1 = 0 & ; \text{at } t = 0 \end{array} \right\} \tag{12}$$

Multiplying both the sides of (8) by $e^{-pt}$ and integrating between the limits 0 to $\infty$, we have the Laplace transform of (8) as :

$$p \, \bar{u}_1 = \nu \frac{\partial^2 \bar{u}_1}{\partial y^2}, \tag{13}$$

and transformed boundary conditions are

$$\left.\begin{aligned} \bar{u}_1 &= \frac{V}{p-in} \text{ at } y = 0\,, \\ \text{and} \qquad \bar{u}_1 &= 0 \qquad \text{at } y = y_0\,. \end{aligned}\right\} \tag{14}$$

Now the solution of (13) using (14) is

$$\bar{u}_1 = \frac{V}{p-in} \quad \frac{\sinh\left[\sqrt{(p/\nu)}\,(y_0 - y)\right]}{\sinh\left[\sqrt{(p/\nu)}\,y_0\right]}\,, \tag{15}$$

The inverse Laplace integral of (15) is evaluated by transforming the path of integration into a closed contour and applying the calculus of residue[11], we obtain

$$u_1 = V\,e^{int} \;\frac{\sinh\left[\sqrt{(n/\nu)}\,(y_0 - y)\right]}{\sinh\left(\sqrt{(in/\nu)}\,y_0\right)} +$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k\, V\, 2\,\nu\, k\,\pi \exp\left(-\nu k^2\,\pi^2\, t/y^2_0\right) \sinh k\pi\,\left(1 - y/y_0\right)}{\left(in\, y^2_0 + \nu\, k^2\,\pi^2\right)}\,. \tag{16}$$

Collecting the real part of equation (16) and from equation (10), equation (5) becomes

$$u = U\,(y/y_0) + V\,\epsilon\,\Big[\cosh\alpha\,\sin\alpha\,\Big\{\cosh\,\alpha\,(1 - y/y_0)\,\sin\alpha\,(1 - y/y_0)\,\times$$

$$\times \cos nt + \sinh\alpha\,(1 - y/y_0)\cos\alpha\,(1 - y/y_0)\sin nt\Big\} +$$

$$+ \sinh\alpha\,\cos\alpha\,\Big\{\sinh\alpha\,(1 - y/y_0)\,\cos\alpha\,(1 - y/y_0)\,\cos nt -$$

$$- \cosh\alpha\,(1 - y/y_0)\,\sin\alpha\,(1 - y/y_0)\,\sin\,nt\Big\}\Big] \Big/ \Big[\cosh^2\alpha\,\sin^2\alpha\, +$$

$$+ \sinh^2\alpha\,\cos^2\alpha\Big] +$$

$$+ V\,\epsilon\left[\sum_{k=1}^{\infty} \frac{(-1)^k\, 2k^3\,\pi^3\,\exp\left(-\nu k^2\,\pi^2\, nt/ny^2_0\right)\sin k\,\pi\,\left(1 - y/y_0\right)}{\left(k^4\,\pi^4 + 4\,\alpha^4\right)}\right], \tag{17}$$

where $\alpha = \sqrt{(n/2\nu)}\,y_0\,.$

The following two particular cases may be derived from (17) :

(*i*) When $y_0 \to \infty$ and $U = 0$, equation (17) is reduced to

$$\frac{u}{V} = \epsilon e^{-\eta} \cos(nt - \eta)\,, \tag{18}$$

where $\eta = y\sqrt{n}\,/\,\sqrt{(2\nu)}.$

This is well known flow caused by an infinite plate executing simple harmonic motion known as Stokes first problem.

(*ii*) For $V=0$, from (17), we obtain

$$\frac{u}{U} = \frac{y}{y_0}\,, \tag{19}$$

which is well-known plane Couette flow.

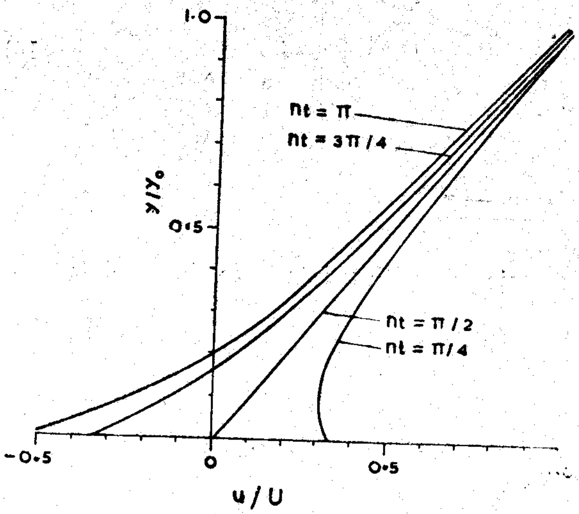


Fig. 1—Velocity profiles plotted against $Y/Y_0$, for $a = 4$ and $\epsilon = 0{\cdot}5$.

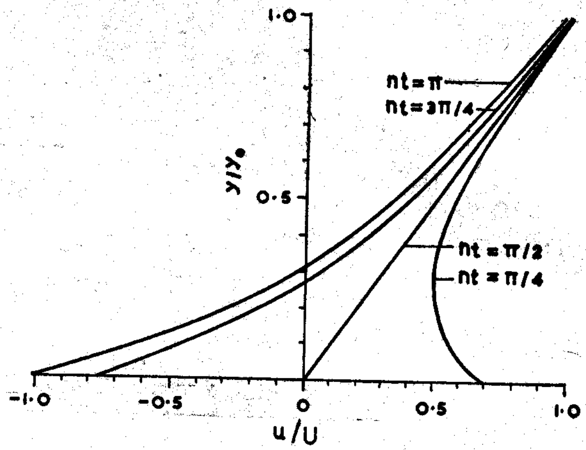


Fig. 2—Velocity profiles plotted against $Y/Y_0$ for $a = 4$ and $\epsilon = 1$.

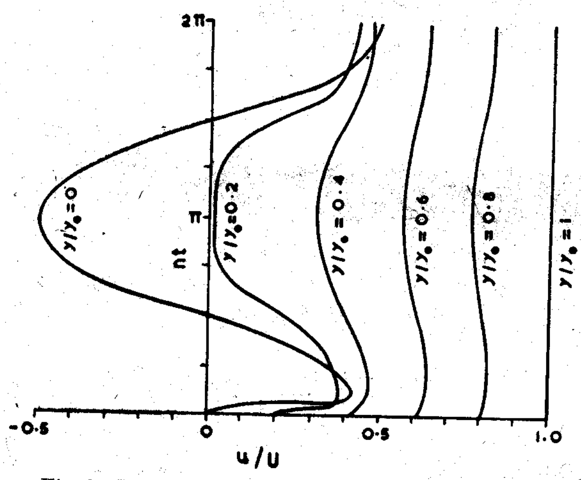


Fig. 3—Velocity profiles plotted against $nt$, for $a = 4$ and $\epsilon = 0{\cdot}5$.

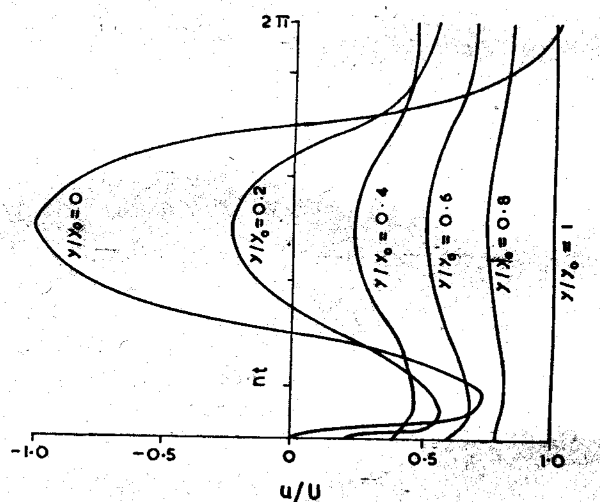


Fig. 4—Velocity profiles plotted against $nt$, for $a = 4$ and $\epsilon = 1$.

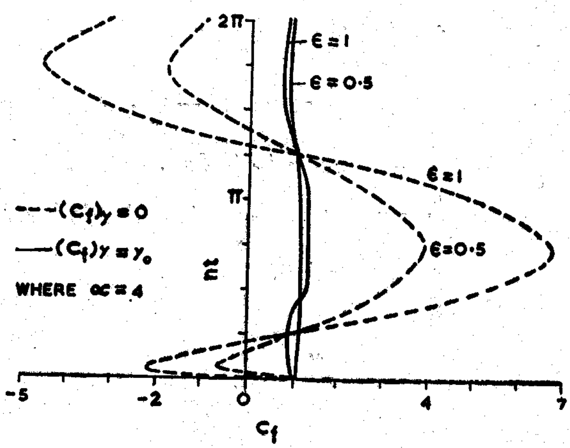


Fig. 5—Coefficient of skin-friction plotted against $nt$.

## COEFFICIENT OF SKIN-FRICTION

Coefficient of skin friction is given by (for $U=V$)

$$C_f = \frac{\tau}{\mu U / y_0} = \frac{y_0}{U} \frac{\partial u}{\partial y} . \tag{20}$$

Thus for the lower plate, we have

$$(C_f)_{y=0} = 1 + \epsilon [\cosh \alpha \sin \alpha \,(Q \sin nt - P \cos nt) + \sinh \alpha \cos \alpha \,(Q \cos nt + P \sin nt)] / [\cosh^2 \alpha \sin^2 \alpha + \sinh^2 \alpha \cos^2 \alpha] - \epsilon \sum_{k=1}^{\infty} \frac{2k^4 \pi^4 \exp(-\nu k^2 \pi^2 t / y^2_0)}{(k^4 \pi^4 + 4 \alpha^4)} , \tag{21}$$

where

$$P = \alpha [\cosh \alpha \cos \alpha + \sinh \alpha \sin \alpha]$$

and

$$Q = \alpha [\sinh \alpha \sin \alpha - \cosh \alpha \cos \alpha] .$$

Similarly for the upper plate, we have

$$(C_f)_{y=y_0} = 1 + \epsilon \alpha [\sinh \alpha \cos \alpha (\sin nt - \cos nt) - \cosh \alpha \sin \alpha (\sin nt + \cos nt)] / [\cosh^2 \alpha \sin^2 \alpha + \sinh^2 \alpha \cos^2 \alpha] - \epsilon \sum_{k=1}^{\infty} \frac{(-1)^k \, 2k^4 \pi^4 \exp(-\nu k^2 \pi^2 t/y^2_0)}{(k^4 \pi^4 + 4\alpha^4)} \tag{22}$$

## NUMERICAL DISCUSSION

The velocity profiles are drawn in Figs. 1-4 for $V=U$, $\epsilon=0\cdot5$ and $\epsilon=1$. The behaviour of the velocity along the plate for various values of $nt$ are shown in Figs. 1 and 2. The oscillating plate sets up a corresponding oscillation in the fluid which is effected by the motion of the upper plate. As expected the resulting motion is the superimposition of the motion due to oscillating plate and the Couette flow. The velocity profiles are plotted against $nt$ for various values of $(y/y_0)$ in Figs. 3 and 4. The oscillations set up in the fluid in the neighbourhood of the oscillating plate are damped as we move towards the upper plate which is moving with a uniform velocity. The coefficient of skin-friction at both the plates is plotted against $nt$ in Fig. 5. For the lower plate, the coefficient of friction is maximum for $nt=3\pi/4$; and for $nt=0$, it gives the coefficient of friction for plane Couette flow. For the upper plate the coefficient of friction oscillates about 1.

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