NOTE ON THE PULSATING FLOW OF n-INCOMPRESSIBLE AND IMMISCIBLE RAREFIED GASES BETWEEN TWO PLATES

RAVI KANT

Agra College, Agra—282002

(*Received* 20 *December* 1977)

An investigation is made to study the unsteady viscous flow of n-immiscible and incompressible rarefied gases, occupying equal heights between parallel and stationary plates under the influence of a periodic pressure gradient superposed on the steady laminar flow. Expressions for velocity distributions have been obtained in exact form. The effects of the rarefaction parameter on the velocity distribution have been shown graphically when there are only two gases.

In the present era of high altitude flights the problems concerning the flow of rarefied gases have been recognised to be of immense importance. For rarefied gases, the ordinary continuum approach fails to yield satisfactory results. However, when the gas is only slightly rarefied, results agreeing with the observed physical phenomena can be obtained by solving the usual Navier-Stokes equations together with modified boundary conditions allowing for a velocity slip and temperature jump at the surface. This scheme of theoretical investigation is particularly suitable for studying the effects of gas rarefaction on any classical viscous flow problem. The problems concerning the flow of immiscible fluids play important roles in medicine, industry and defence.

The problems of immiscible fluids under the influence of a constant or periodic time dependent pressure gradient have been studied by Bird[1] *et al.*, Kapur & Shukla [2, 3], Gupta & Goyal[4]. In all these problems the flow of normal density fluids was considered. It is of interest to study how Gupta & Goyal[4]'s results get modified when his no-slip boundary conditions are replaced by the velocity slip conditions. This indeed is the motivation for the present investigation.

In this paper, we have studied the viscous unsteady flow of n-incompressible and immiscible fluids occupying equal heights between two parallel and stationary plates under a periodic time-dependent pressure gradient superposed on the steady laminar flow in slip flow regime. Expressions for velocity distributions have been obtained in exact form for all the gases. Comparison of results of slip and no-slip cases is shown graphically in the case of two fluids in particular. The assumption that the fluid is incompressible is a fairly valid approximation at least for low mach number flows. However, for a more accurate description of the flow in rarefied gases, the compressibility effects should be taken into account.

MATHEMATICAL FORMULATION OF THE PROBLEM

Consider the flow of n-viscous incompressible and immiscible rarefied gases each occupying a height h filling the gap between two infinite parallel and stationary plates kept at a distance nh apart. Choose a cartesian co-ordinate system with x-axis along the lower plate and parallel to the flow, y-axis perpendicular to it and upwards and z-axis lying on the plate.

The governing equations of motion for viscous incompressible and immiscible fluids (rarefied gases) neglecting external forces are

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\rho_j} \cdot \frac{\partial p}{\partial x} + \nu_j \frac{\partial^2 u_j}{\partial y^2} \tag{1}$$

$$0 = \frac{\partial p}{\partial y} \tag{2}$$

where u_j, ρ_j, ν_j are the velocity, density and kinematic viscosity of the j-th gas starting from lower to the upper plate.

Equation (2) reveals that the pressure is constant across the section of the channel.

The boundary conditions are

$$\left.\begin{aligned} u_1(0,t) &= L_1 \frac{\partial u_1}{\partial y} \\ u_n(nh,t) &= L_1 \frac{\partial u_n}{\partial y} \end{aligned}\right\} \tag{3}$$

where

$$L_1 = \left(\frac{2-f}{f}\right)$$

L being mean free path and f being the Maxwell's reflecxion coefficient.

SOLUTION OF THE MORE PROBLEM

The pressure gradient becomes only function of t for the uniform pulsating flow for which $\frac{\partial u_j}{\partial x} = 0$, let us express it by the following Fourier series

$$-\frac{1}{\rho_j}\frac{\partial p}{\partial x} = X_0 + Re \sum_{m=1}^{\infty} X_m \, e^{imt} \tag{4}$$

where

$$X_m = X_{cm} - iX_{sm}$$

and X_{cm} and X_{sm} are constants which represent the amplitudes of elemental vibrations.

Similarly, let us express the longitudinal velocity as

$$u_j = u_o^{(j)} + Re \sum_{m=1}^{\infty} u_m^{(j)} \, e^{imt} \tag{5}$$

where

$$u_m^{(j)} = u_{cm}^{(j)} - i u_{sm}^{(j)}$$

and the coefficients $u_m(j)$, $u_{cm}(j)$ and $u_{sm}(j)$ are the functions of y only.

Substituting (4) and (5) in (1) and equating the terms of the same family, we obtain

$$\frac{d^2 u_0^{(j)}}{dy^2} + \frac{X_0}{\nu_j} = 0 \tag{6}$$

$$\frac{d^2 u_m(j)}{dy^2} - \frac{im}{\nu_j} u_m(j) + \frac{X_m}{\nu_j} = 0 \tag{7}$$

The boundary conditions reduce to

$$\left.\begin{aligned} u_0^{(1)} &= L_1 \frac{du_0^{(1)}}{dy}, \; u_m^{(1)} = L_1 \frac{du_m^{(1)}}{dy} \quad \text{at } y = 0 \\ u_0^{(n)} &= -L_1 \frac{du_0^{(n)}}{dy}, \; u_m^{(n)} = -L_1 \frac{du_m^{(n)}}{dy} \quad \text{at } y = nh \end{aligned}\right\} \tag{8}$$

It is assumed that for moderate pulsation, the shape of the interfaces does not change. Neglecting the surface tension at the interfaces, the velocities at the interfaces can be taken as

$$u_j = A_p + Re \sum_{m=1}^{\infty} (Am)_p \, e^{imt} \quad \text{on } y = ph \tag{9}$$

$$(j = p, p+1 \text{ and } p = 1, 2, \ldots\ldots, (n-1)).$$

where

A_p and $(Am)_p$ are constants to be determined.

With the help of (5) and (9), we obtain

$$u_0^{(p)} = u_0^{(p+1)} = A_p \quad \text{on } y = ph \tag{10}$$

and

$$u_m^{(p)} = u_m^{(p+1)} = (Am)_p \quad \text{on } y = ph \tag{11}$$

The solution of the equation (6) is

$$u_0^{(j)} = -1/2 \frac{X_0}{\nu_j} y^2 + B_0^{(j)} y + C_0^{(j)} \tag{12}$$

This equation shows that the steady part of velocity distribution is parabolic.

From (8) and (10), we obtain

$$B_0^{(1)} = \frac{1}{(L_1 + h)} \left(A_1 + \frac{1}{2} \frac{X_0 h^2}{\nu_1} \right) \tag{13}$$

$$C_0^{(1)} = \frac{L_1}{(L_1 + h)} \left(A_1 + \frac{1}{2} \frac{X_0 h^2}{\nu_1} \right) \tag{14}$$

....

$$B_0^{(p)} = -\frac{1}{h} (A_{p-1} - A_p) + \frac{1}{2} \frac{X_0 h}{\nu_p} (2p - 1) \tag{15}$$

$$C_0^{(p)} = p A_{p-1} - (p-1) A_p - \frac{1}{2} \frac{X_0 h^2}{\nu_p} p (p - 1) \tag{16}$$

....

$$B_0^{(n)} = \frac{1}{(L_1 + h)} \left[\frac{X_0}{2\nu_n} \{ 2L_1 nh + (2n - 1) h^2 \} - A_{n-1} \right] \tag{17}$$

$$C_0^{(n)} = \frac{1}{(L_1 + h)} \left[(L_1 + nh) A_{n-1} - \frac{X_0 h^2 (n-1)}{2\nu_n} \left\{ L_1 (n + 1) + nh \right\} \right] \tag{18}$$

Similarly, the solution of the equation (7) is

$$u_m^{(j)} = B_j e^{s_j y} + C_j e^{-s_j y} + \frac{iXM}{m} \tag{19}$$

where

$$S_j = \sqrt{\frac{im}{\nu_j}} \text{ and } M = -\frac{iX_m}{m}$$

From (8) and (11), we obtain

$$B_1 = \frac{L_1 S_1 (1 + \{(Am)_1 - M\} + Me^{-S_1 h}}{2 \{\sinh (S_1 h) + L_1 S_1 \cosh (S_1 h)\}} \tag{20}$$

$$C_1 = \frac{(L_1 S_1 - 1)\{(Am)_1 - M\} - Me S_1^h}{2 \{\sinh (S_1 h) + L_1 S_1 \cosh (S_1 h)\}} \tag{21}$$

$$B_p = \frac{M\{- (Am)_{p-1} + \{(Am)_p - M\} e S^{ph}}{2 \sinh S_p h \; e S^{ph}} \tag{22}$$

$$C_p = \frac{(Am)_{p-1}{}^{-M} + \{M - (Am)_p\}\, e - S_p{}^h}{2 \sinh (S_p h)\, e - S_p{}^{ph}} \tag{23}$$

....

$$B_n = \frac{L_1 S_n - 1)\{(Am)_{n-1} - M\} + Me\, S_n{}^h}{2\, e S_n{}^h \{\sinh (S_n h) + L_1 S_n \cosh (S_n h)\}} \tag{24}$$

$$C_n = \frac{(1 + L_1 S_n)\{(Am)_{n-1} - M\} + M\, e - S_n{}^h}{2\, e - S_n{}^{nh} \{\sinh (S_n h) + L_1 S_n \cosh (S_n h)\}} \tag{25}$$

Hence Using (5), (12) and (19), the complete velocity distribution is given by

$$u_j = -\frac{1}{2} \frac{X_0}{\nu_j} y^2 + B_0{}^{(j)} y + C_0{}^{(j)} +$$

$$+ Re \sum_{m=1}^{\infty} e^{imt} \left\{ B_j e^{S_j y} + C_j e^{-S_j y} + M \right\} \tag{26}$$

DETERMINATION OF INTERFACE VELOCITIES

In order to determine the interface velocities, the continuity of the shear at the interface is to be considered.

$$\left(\mu_p \frac{\partial u_p}{\partial y} \right)_{y=ph} = \left(\mu_{p+1} \frac{\partial u_{p+1}}{\partial y} \right)_{y=ph}, \ (p\text{ - }th \text{ interface}) \tag{27}$$

Substituting (26) in (27), using (13) to (18) and comparing the terms of the same family, we have

$$\left(\frac{\mu_1}{L_1 + h} + \frac{\mu_2}{h} \right) A_1 - \frac{\mu_2 A_2}{h} = \frac{X_0 h}{2} \left[\frac{\rho_1 (2L_1 + h)}{(L_1 + h)} + \rho_2 \right] \tag{28}$$

....

$$-\frac{\mu_p}{h} A_{p-1} + \frac{\mu_p + \mu_{p+1}}{h} A_p - \frac{\mu_{p+1}}{h} A_{p+1} = \frac{X_0 h}{2} \left(\rho_p + \rho_{p+1} \right) \tag{29}$$

....

$$-\frac{\mu_{n-1}}{h} A_{n-2} + \left(\frac{\mu_{n-1}}{h} + \frac{\mu_n}{L_1 + h} \right) A_{n-1} = \frac{X_0 h}{2} \left[\rho_{n-1} + \frac{(2L_1 + h}{(L_1 + h)} \rho_n \right] \tag{30}$$

From the equations (28) to (30), we can determine A_p. Similarly, $(Am)_p$ can be obtained using equations (20) to (25).

Skin Friction

The skin friction at the lower and upper plates are given by

$$\tau_0 = \frac{\mu_1}{L_1} \left[C_0{}^{(1)} + Re \sum_{m=1}^{\infty} e^{imt} (B_1 + C_1 + M) \right] \tag{31}$$

$$\tau_n = -\frac{\mu_n}{L_1} \left[-\frac{1}{2} \frac{n^2 h^2 X_0}{\nu_n} + nh\, B_0{}^{(n)} + C_0{}^{(n)} + \right.$$

$$\left. + Re \sum_{m=1}^{\infty} e^{imt} \left(B_n e^{S_n{}^{nh}} + C_n e^{-S_n{}^{nh}} + M \right) \right. \tag{32}$$

Particular Case : *Two Ratefied Gases*

In this case there are only two rarefied gases filling the gap equally between two plates at a distance $2h$ apart.

The velocities of the lower and upper rarefied gases are given by

$$u_j = -\frac{1}{2}\frac{X_0}{\nu_j} y^2 + B_0^{(j)} y + C_0^{(j)} +$$

$$+ Re \sum_{m=1}^{\infty} e^{imt} \left\{ (B_j e^{S_j y} + C_j e^{-S_j y} + M \right\} \tag{33}$$

$$(j = 1, 2)$$

where $j=1$ for the lower gas occupying the region $0 \leqslant y \leqslant h$; $j=2$ for the upper gas occupying the region $h_1\, y \leqslant 2h$ and $B_0^{(1)}$, $C_0^{(1)}$, B_1 and C_1 are given by (13), (14), (20) and (21) respectively and $B_0^{(2)}$, $C_0^{(2)}$, B_2 and C_2 can be obtained from (17), (18), (24) and (25) respectively on putting $n=2$.

The velocity at the common interface is given by

$$u = A_1 + Re \sum_{m=1}^{\infty} (Am)_1\, e^{imt} \tag{34}$$

where

$$A_1 = \frac{X_0 h\,(2L_1 + h)\,(\rho_1 + \rho_2)}{2(\mu_1 + \mu_2)} \tag{35}$$

and

$$(Am)_1 \left[\frac{\mu_1 S_1 \{ \cosh (S_1 h) + L_1 S_1 \sinh (S_1 h) \}}{\sinh (S_1 h) + L_1 S_1 \cosh (S_1 h)} + \frac{\mu_2 S_2 \{ \cosh (S_2 h) + L_1 S_2 \sinh (S_2 h) \}}{\sinh (S_2 h) + L_1 S_2 \cosh (S_2 h)} \right]$$

$$= M \left[\frac{\mu_1 S_1 \{ \cosh (S_1 h) + L_1 S_1 \sinh (S_1 h) - 1 \}}{\sinh (S_1 h) + L_1 S_1 \cosh (S_1 h)} + \frac{\mu_2 S_2 \{ \cosh (S_2 h) + L_1 S_2 \sinh (S_2 h) - 1 \}}{\sinh (S_2 h) + L_1 S_2 \cosh (S_2 h)} \right] \tag{36}$$

The skin friction at lower and upper plates are given by

$$\tau_0 = \frac{\mu_1}{L_1} \left[C_0^{(1)} + Re \sum_{m=1}^{\infty} (B_1 + C_1 + M)\, e^{imt} \right] \tag{37}$$

and

$$\tau_2 = -\frac{\mu_2}{L_1} \left[-\frac{2X_0 h^2}{\nu_2} + 2\, B_0^{(2)}\, h + C_0^{(2)} + \right.$$

$$\left. + Re \sum_{m=1}^{\infty} e^{imt} \left((B_2 e^{2 S_2 h} + C_2 e^{-2 S_2 h} + M \right) \right] \tag{38}$$

DISCUSSION

The velocity profiles for the flow of two immiscible and incompressible rarefied gases occupying a height h each have been drawn for different values of rarefaction parameter $\xi = \frac{L_1}{h}$ with axial pressure gradient $\frac{X_0 h^3}{\nu^2} = 1$ for the steady case. There are three sets of profiles having three each. In one set the kinematic viscosity is the same for both the gases. In this case the velocity profiles are parabolic. In other sets the kinematic viscosity of the upper gas is lesser or greater than that of lower one. When the kinematic viscosity of the upper gas is lesser than that of lower one, the velocity is maximum in the upper portion ($h \leqslant y \leqslant 2h$), while the kinematic viscosity of the upper gas is greater than that of lower one, it is maximum in the lower portion ($0 \leqslant y \leqslant h$). From the figure the parabolic profiles for the gas of the same viscosity can be compared with the velocity profiles for the gases, filling the upper portion, whose kinematic viscosity is lesser or greater than that of lower one. It is also observed that the magnitude of

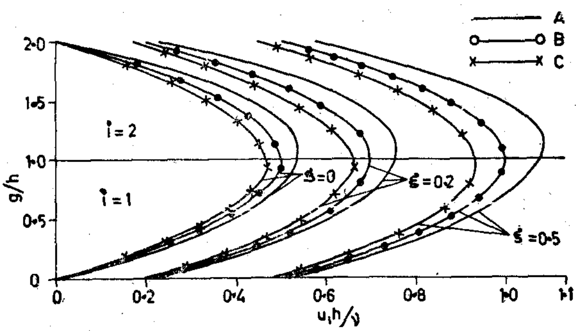

Fig. 1—Distribution of velocity for two rarefied gases occupying a flight h each.
A for $\mu_1=\mu$, $\rho_1=\rho$, $\mu_2=\cdot 3\mu$, $\rho_2=\rho\cdot\rho$
B for $\mu_1=\mu$, $\rho_1=\rho$, $\mu_2=\cdot 6\mu$, $\rho_2=\cdot 5\rho$ &
C for $\mu_1=\mu_2=\mu$, $\rho_1=\rho_2=\rho$.

the velocity depends on the kinematic viscosity of the gas. It increases for the decrease of ν and vice-versa for constant values of rarefaction parameter and pressure gradient. The figure also exhibits the effect of the rarefaction parameter on the velocity field. We observe that an increase in the rarefaction parameter increases the velocity at any point of the rarefied gases. It is noted from equations (13), (14), (17), (18), (33) and (34) that the velocity at any point of the gases is multiplied by the same quantity by which the pressure gradient is multiplied which is in agreement with the physical situation.

ACKNOWLEDGEMENTS

The author wishes to offer his sincere thanks to Dr. G.C. Sharma for his guidance in the preparation of this research paper. He is also thankful to C.S.I.R. for the award of a fellowship.

REFERENCES

1. BIRD, R.B., STEWART, W.E. & LIGHTFOOD, E.N., 'Transport Phenomena' (John Wiley & Sons, New York), 1960.
2. KAPUR, J.N. & SHUKLA, J.B., *Appl. Scient. Res.*, **13** (1963), 55-60.
3. KAPUR, J.N. & SHUKLA, J.B., *ZAMM*, **44** (1964), 268.
4. GUPTA, M.C. & GOYAL, M.C., *Indian J. Pure & Appl. Maths.*, **4** (1973), 680-695.