BENDING OF AELOTROPIC BLOCKS — II

S. RAM RAO

Science College, Saifabad, Hyderabad

&

G. LAKSHMINARAYANA

Nagarjunasagar Engineering College, Hyderabad

(*Received* 1 *January* 1972)

In Part I, the authors had obtained a solution for bending an aelotropic circular block into an ellipsoidal shell and obtained the solution for the problem of bending an aelotropic circular block into a spherical shell as a particular case. In this paper, the solution for bending a circular block into a paraboloidal shell has been obtained on the same lines in terms of a completely general strain energy function for both compressible and incompressible materials.

The theory of finite deformation received fresh impetus when Rivlin[1] obtained exact solutions for a number of problems specially for incompressible bodies, in terms of an arbitrary strain energy function. References to various developments are found in the surveys by Rivlin[1], Truesdell & Toupin[2], Green & Zerna[3], Green & Adkins[4], and Eringen[5]. Recently, Green & Adkins[6] examined the finite flexure of an aelotropic cuboid. The problem of bending an aelotropic circular block into an ellipsoidal shell was considered by the present authors[7] in Part I and a solution was obtained in terms of a completely general strain energy function. The solution for the problem of bending a circular block into a spherical shell was obtained as a particular case. In this paper, the problem of bending a circular block into a paraboloidal shell has been considered. The solution has been obtained in terms of a completely general strain energy function for both compressible and incompressible materials.

NOTATION AND FORMULAE

We adopt the notation and formulae of Green & Adkins[6]. The strain energy W of a homogeneous aelotropic body is expressed as a polynomial

$$W = W(e_{ij}) \tag{1}$$

in the components of strain e_{ij}. The stress tensor T^{ij} for a compressible body is given by

$$T^{ij} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial e_{rs}} + \frac{\partial W}{\partial e_{sr}}\right)\frac{\partial \theta^i}{\partial x^r}\,\frac{\partial \theta^j}{\partial x^s}, \tag{2}$$

where

$$I_3 = \left| 2\,e_{rs} + \delta_{rs} \right| \tag{3}$$

For an incompressible body, $I_3 = 1$, and the stress tensor T^{ij} is given by

$$T^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial e_{rs}} + \frac{\partial W}{\partial e_{sr}}\right)\frac{\partial \theta^i}{\partial x^r}\,\frac{\partial \theta^j}{\partial x^s} + pG^{ij}, \tag{4}$$

where p is a scalar function of θ^i, and G^{ij} is the contravariant metric tensor of the curvilinear coordinates θ^i in the deformed body.

The equations of equilibrium, in the absence of body forces, are

$$T^{ij}\Big|_j = 0\,. \tag{5}$$

BENDING OF A CIRCULAR BLOCK

Let us consider bending of an aelotropic compressible circular block into a paraboloidal shell. Suppose that a circular block in the undeformed state is bounded by the planes $x_3 = a_1$, $x_3 = a_2$, $(a_2 > a_1)$, and the cylinder $x_1^2 + x_2^2 = a^2$. The block is bent symmetrically about the x_3-axis into a part of a paraboloidal shell, whose inner and outer boundaries are the paraboloids of revolution obtained by revolving the confocal parabolas

$$x_3 = \frac{\xi_i^2 - \eta^2}{2},$$

$$x_1 = \xi_i \eta \, , i = 1 , 2 , \tag{6}$$

about the x_3-axis, and the edge $\eta = \alpha$. Let y_i-axes coincide with x_i-axes, and the curvilinear coordinates θ^i in the deformed state be a system of orthogonal curvilinear coordinates (ξ, η, φ), where is the angle between $y_1\, y_3$-plane and the plane through a point in space and the y_3-axis. Then,

$$y_1 = \xi\, \eta \cos \varphi,\; y_2 = \xi\, \eta \sin \varphi,\; y_3 = (\xi^2 - \eta^2)/2 \tag{7}$$

Since the deformation is symmetric about the x_3-axis, we see that

(i) the planes x_3 = constant in the undeformed state become the paraboloidal surfaces ξ = constant in the deformed state;

(ii) the surfaces $x_1^2 + x_2^2$ = constant in the undeformed state become the surfaces η = constant in the deformed state; and

(iii) $\tan^{-1} (x_2/x_1) = \varphi$.

Thus, the deformation is given by

$$\xi = f(x_3),\; \eta = F(x_1^2 + x_2^2),\; \varphi = \tan^{-1} (x_2/x_1) \tag{8}$$

The strain components are given by

$$\left.\begin{aligned} 2\, e_{11} &= 4\, x_1^2 F'^2 (\xi^2 + \eta^2) + \xi^2 \eta^2 x_2^2/(x_1^2 + x_2^2)^2 - 1 , \\ 2\, e_{22} &= 4\, x_2^2 F'^2 (\xi^2 + \eta^2) + \xi^2 \eta^2 x_1^2/(x_1^2 + x_2^2)^2 - 1 , \\ 2\, e_{33} &= f'^2 (\xi^2 + \eta^2) - 1 , \\ 2\, e_{12} &= 4\, x_1 x_2 F'^2 (\xi^2 + \eta^2) - x_1 x_2 \xi^2 \eta^2/(x_1^2 + x_2^2)^2 ; \\ e_{23} &= e_{31} = 0 . \end{aligned}\right\} \tag{9}$$

The stress tensor (2) for compressible material has components

$$\left.\begin{aligned} T^{11} &= \frac{f'^2}{\sqrt{I_3}} \frac{\partial W}{\partial e_{33}} , \\ T^{22} &= \frac{4\, F'^2}{\sqrt{I_3}} \left\{ x_1^2 \frac{\partial W}{\partial e_{11}} + x_2^2 \frac{\partial W}{\partial e_{22}} + x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \\ T^{33} &= \frac{1}{(x_1^2 + x_2^2)^2 \sqrt{I_3}} \left\{ x_2^2 \frac{\partial W}{\partial e_{11}} + x_1^2 \frac{\partial W}{\partial e_{22}} + x_1 x_2 \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \\ T^{23} &= \frac{F'}{2\,(x_1^2 + x_2^2) \sqrt{I_3}} \left\{ 2\, x_1 x_2 \left(\frac{\partial W}{\partial e_{22}} - \frac{\partial W}{\partial e_{11}} \right) + (x_1^2 - x_2^2) \left(\frac{\partial W}{\partial e_{12}} + \frac{\partial W}{\partial e_{21}} \right) \right\} \\ T^{12} &= T^{31} = 0 , \end{aligned}\right\} \tag{10}$$

where

$$I_3 = 4 f'^2 F'^2 (\xi^2 + \eta^2)^2 \xi^2 \eta^2 \tag{11}$$

The metric tensor for the strained state of the body is given by

$$G_{ij} = \begin{bmatrix} \xi^2 + \eta^2 & 0 & 0 \\ 0 & \xi^2 + \eta^2 & 0 \\ 0 & 0 & \xi^2\eta^2 \end{bmatrix} \tag{12}$$

The equations of equilibrium (5) reduce to

$$\left.\begin{aligned} & T^{11}{}_{,1} + (2\,\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\,T^{11} + \Gamma^1_{22}\,T^{22} + \Gamma^1_{33}\,T^{33} = 0\,, \\ & T^{22}{}_{,2} + (\Gamma^1_{12} + 2\,\Gamma^2_{22} + \Gamma^3_{32})\,T^{22} + \Gamma^2_{11}\,T^{11} + \Gamma^2_{33}\,T^{33} = 0\,, \\ & T^{23}{}_{,2} + (\Gamma^1_{12} + \Gamma^2_{22} + 3\,\Gamma^3_{32})\,T^{23} = 0\,. \end{aligned}\right\} \tag{13}$$

The above equations of equilibrium, as they stand, do not seem to admit a solution. However, a solution can be obtained if we assume η to be so small[8] that η^2 may be neglected when compared with η, and that $\eta = F(x_1^2 + x_2^2) = K\,(x_1^2 + x_2^2)^{1/2}$. Physically this implies that the paraboloidal shell is within that paraboloidal shell for which η = constant. It is further noted that the deformed shell is shallow or deep according as ξ_i, $i = 1, 2$, are large or small respectively, where ξ_i, $i = 1, 2$, give the boundaries of the deformed shell. Then the expressions corresponding to (9) to (12) reduce to

$$\left.\begin{aligned} 2\,e_{11} &= K^2\,\xi^2 - 1\,,\ 2\,e_{22} = K^2\,\xi^2 - 1\,, \\ 2\,e_{33} &= f'^2\,\xi^2 - 1\,,\ e_{12} = e_{23} = e_{31} = 0\,, \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} T^{11} &= \frac{f'^2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial e_{33}}\,, \quad T^{22} = \frac{K^2}{\sqrt{I_3}} \cdot \frac{\partial W}{\partial e_{11}}\,, \\ T^{33} &= \frac{K^2}{\eta^2\sqrt{I_3}} \cdot \frac{\partial W}{\partial e_{11}}\,, \quad T^{12} = T^{31} = T^{23} = 0\,, \end{aligned}\right\} \tag{15}$$

$$I_3 = K^4 f'^2\,\xi^6 \tag{16}$$

$$G_{ij} = \begin{bmatrix} \xi^2 & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \eta^2\,\xi^2 \end{bmatrix}. \tag{17}$$

Then, the second and the third equations of equilibrium (13) are satisfied identically. The first equation gives

$$f'^2\,\xi^2 \frac{\partial}{\partial \xi}\left(\frac{\partial W}{\partial e_{33}}\right) + f'^2\,\xi\left(\frac{\partial W}{\partial e_{33}}\right) + f''\,\xi^2\left(\frac{\partial W}{\partial e_{33}}\right) - 2\,K^2\,\xi\left(\frac{\partial W}{\partial e_{11}}\right) = 0 \tag{18}$$

Now

$$\begin{aligned} \frac{\partial W}{\partial \xi} &= \frac{\partial W}{\partial e_{11}}\,\frac{\partial e_{11}}{\partial \xi} + \frac{\partial W}{\partial e_{22}}\,\frac{\partial e_{22}}{\partial \xi} + \frac{\partial W}{\partial e_{33}}\,\frac{\partial e_{33}}{\partial \xi} \\ &= f'^2\,\xi\,\frac{\partial W}{\partial e_{33}} + f''\,\xi^2\,\frac{\partial W}{\partial e_{33}} + 2\,K^2\,\xi\,\frac{\partial W}{\partial e_{11}} \end{aligned} \tag{19}$$

From (18) and (19), we get

$$\frac{\partial W}{\partial \xi} = \frac{\partial}{\partial \xi}\left(f'^2\,\xi^2\,\frac{\partial W}{\partial e_{33}}\right), \tag{20}$$

which on integration gives

$$W = f'^2\,\xi^2\left(\frac{\partial W}{\partial e_{33}}\right) - W_0\,, \tag{21}$$

where W_0 is a constant.

This gives

$$f'^2 = \frac{W + W_0}{\xi^2 \dfrac{\partial W}{\partial e_{33}}} \tag{22}$$

The non-vanishing physical components of stress, from (15) to (17) are given by

$$\left.\begin{aligned} \sigma_{11} &= \frac{1}{K^2 \xi^2} \sqrt{(W + W_0) \frac{\partial W}{\partial e_{33}}} \\ \sigma_{22} = \sigma_{33} &= \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ (W + W_0)} \end{aligned}\right\} \tag{23}$$

BOUNDARY CONDITIONS

I. If $-R_i$, $(i = 1, 2)$, are the applied normal tractions on the inner and the outer surfaces of the shell, we have

$$\sigma_{11} = -R_i \quad \text{when} \quad \xi = \xi_i, \quad i = 1, 2. \tag{24}$$

which on substitution in (23) gives

$$\frac{1}{K^4 \xi^4} \left\{ W(\xi_i) + W_0 \right\} \left(\frac{\partial W}{\partial e_{33}} \right)_{\xi = \xi_i} = R_i^2, \qquad i = 1, 2. \tag{25}$$

Solving these, we get the values of the constants W_0 and K.

On the edge $\eta = \alpha$, the distributions of tractions per unit arc between φ and $\varphi + d\varphi$ give rise to a force F_1 and a couple M_1 about the origin given by

$$\left.\begin{aligned} F_1 &= \alpha \int_{\xi_1}^{\xi_2} \xi^2 \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ (W + W_0)}\, d\xi \\ M_1 &= -\frac{\alpha}{2} \int_{\xi_1}^{\xi_2} \xi^4 \frac{\partial W}{\partial e_{11}} \sqrt{\frac{\partial W}{\partial e_{33}} \Big/ (W + W_0)}\, d\xi \end{aligned}\right\} \tag{26}$$

When the material is incompressible, $I_3 = 1$. Then from (16) we have

$$\frac{df}{dx_3} = \frac{1}{K^2 \xi^3},$$

which on integration gives

$$x_3 = \frac{K^2 \xi^4}{4} + B, \tag{27}$$

where B is an arbitrary constant.

As the internal and the external boundaries of the paraboloidal shell are given by $\xi = \xi_i$, $i = 1, 2$, respectively which were initially the planes $x_3 = a_1$ and $x_3 = a_2$, equation (27) gives

$$\frac{1}{K^2} = \frac{\xi_1^2 - \xi_2^2}{4\,(a_1 - a_2)},$$

and

$$B = \frac{a_1\,\xi_2^4 - a_2\,\xi_1^4}{\xi_2^4 - \xi_1^4} \tag{28}$$

The equations of equilibrium in this case, in view of the components of stress tensor (4), give

$$\frac{\partial p}{\partial \xi} + \xi^2 \frac{\partial}{\partial \xi}\left(f'^2 \frac{\partial W}{\partial e_{33}}\right) + 4\,\xi f'^2 \left(\frac{\partial W}{\partial e_{33}}\right) - 2\,K^2 \xi \left(\frac{\partial W}{\partial e_{11}}\right) = 0, \tag{29}$$

$$\frac{\partial p}{\partial \eta} = 0, \quad \frac{\partial p}{\partial \varphi} = 0. \tag{30}$$

The equations (30) show that p is a function of ξ alone.

Equation (29) in view of (19) gives

$$p = W + W_0 - \frac{1}{K^4\,\xi^4}\,\frac{\partial W}{\partial e_{33}} \tag{31}$$

Then the physical components of stress are given by

$$\sigma_{11} = W + W_0,$$

$$\sigma_{22} = \sigma_{33} = W + W_0 + K^2\,\xi^2\,\frac{\partial W}{\partial e_{11}} - \frac{1}{K^4\,\xi^4}\cdot\frac{\partial W}{\partial e_{33}} \tag{32}$$

II. If the inner boundary $\xi = \xi_1$ of the shell is free from tractions, we must have

$$\sigma_{11} = 0 \text{ when } \xi = \xi_1,$$

which on substitution in (32) gives

$$W_0 = -W(\xi_1) \tag{33}$$

On the outer boundary $\xi = \xi_2$, we have to apply a normal traction R given by

$$R = \sigma_{11}(\xi_2) = W(\xi_2) - W(\xi_1) \tag{34}$$

On the edge $\eta = \alpha$, the distribution of tractions between φ and $\varphi + d\varphi$ give rise to a force F_2 and a couple M_2 about the origin which are given by

$$F_2 = \alpha \int_{\xi_1}^{\xi_2} \xi^2 \left[W + W_0 + K^2\,\xi^2 \cdot \frac{\partial W}{\partial e_{33}} - \frac{1}{K^4\,\xi^4}\cdot\frac{\partial W}{\partial e_{33}} \right] d\xi,$$

$$M_2 = -\frac{\alpha}{2} \int_{\xi_1}^{\xi_2} \xi^4 \left[W + W_0 + K^2\,\xi^2 \cdot \frac{\partial W}{\partial e_{33}} - \frac{1}{K^4\,\xi^4}\cdot\frac{\partial W}{\partial e_{33}} \right] d\xi. \tag{35}$$

Thus we see that to bend a circular block into a paraboloidal shell, we require a resultant force F and a couple M on the edge together with a normal traction R on the outer surface.

REFERENCES

1. RIVLIN, R. S., "Rheology I", (Academic Press, N.Y.) Edited by F. R. Eirish (1956), p. 351.
2. TRUESDELL, C. & TOUPIN, R. A., "Handbuck der Physik" 3/1, (Springer-Verlog, Berlin) 1960.
3. GREEN, A. E. & ZERNA, W., "Theoretical Elasticity", (Oxford University Press, Oxford) 1954.
4. GREEN, A. E. & ADKINS, J. E., "Large Elastic Deformation and Non-linear Continuum Mechanics", (Oxford University Press, Oxford) 1960.
5. ERINGEN, A. C., "Non-linear Theory of Continuous Media", (McGraw Hill), 1962.

6. GREEN, A. E. & ADKINS, J. E., *Arch. Rat. Mech. Anal.*, **VIII** (1961), 9.

7. RAM RAO, S. & LAKSHMINARAYANA, G., "Bending of Aelotropic Block—I", *Def. Sci. J.*, **23** (1973), 11.

8. LAKSHMINARAYANA, G. & BHIMSAIN RAO, *Proc. Nat. Acad. Sci.*, India **35** A (1965), 133.