LUBRICATION OF A ROTATING CIRCULAR STEP BEARING WITH BINGHAM PLASTIC LUBRICANTS

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The flow of an incompressible isotropic Bingham material between two concentric discs, one rotating about the other, has been investigated on the basis of usual approximations of hydrodynamic theory of lubrication. The lifting force and the moment of friction of the bearing have been calculated. The results have been compared with those obtained by Tarapov for a Newtonian lubricant having the same viscosity value as the plastic fluidity of the Bingham material.

It is well known that a linear relation between stress and rate of deformation exists for Newtonian viscous material as well as for a Bingham material. For the Bingham material, a certain critical stress, called the yield value, has to be exceeded before the flow starts. Assuming that there is flow of the Bingham material throughout the region of the film, the problem of rotating circular step bearing has been investigated. The lifting force of the step bearing has been calculated for two cases (a) when the feeding of the lubricant is done from the centre of the disc to the periphery, (b) when the feeding of the lubricant is done from the periphery to the centre of the disc. Moment of friction acting on the discs has been calculated for different values of Bingham number and for various values of the ratio of the radii of the discs. It is found that the value of the moment of friction is greater for a Bingham plastic lubricant than for a Newtonian lubricant having the same viscosity value as the reciprocal mobility of the Bingham material. It is also found that the moment of friction increases with the increase of Bingham number within the considered range.

EQUATIONS GOVERNING THE FLOW

The flow of an isotropic incompressible Bingham material is given by[1]

$$p_{ik}' = 2\eta' e_{ik} \qquad \left(\tfrac{1}{2}\, p_{ik}'\, p_{ik}' \geqslant \eta_2^2\right)$$

where,

$$\eta' = \eta_1 + \frac{\eta_2}{I^{\frac{1}{2}}}\ ; \quad I = 2e_{ik}'\ e_{ik}'$$

e_{ik} = component of rate of strain tensor

p_{ik} = component of stress tensor

Primes denote the deviatoric components of tensors

η_1 = reciprocal mobility

η_2 = yield value.

Let R_1, R_2, h, p_1 and p_2 represent radius of the inner disc, radius of the outer disc, clearance, inside pressure, and outside pressure respectively. Fig. 1 gives the geometry of the problem.

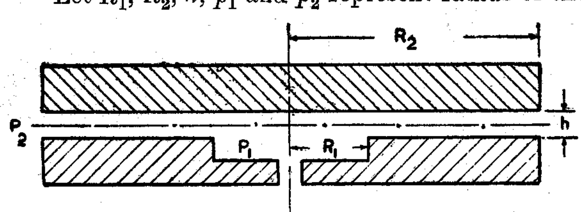

Fig. 1—Geometry of an externally pressurized step bearing.

Let v_r, v_θ, v_z be the radial, circumferential and axial velocities respectively, ρ be the density, and p be the pressure.

The equations governing the steady auxiliary symmetric flow in cylindrical coordinates are given as

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}-\frac{v^2_\theta}{r}\right)=-\frac{\partial p}{\partial r}+\eta_1\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}\right)+$$

$$+\eta_2\left[\frac{\partial}{\partial r}\frac{2(\partial v_r/\partial r)}{I^{\frac{1}{2}}}+\frac{\partial}{\partial z}\frac{(\partial v_r/\partial z)+(\partial v_z/\partial r)}{I^{\frac{1}{2}}}+\frac{2}{rI^{\frac{1}{2}}}\left(\frac{\partial v_r}{\partial r}-\frac{v_r}{r}\right)\right],$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta\, v_r}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=\eta_1\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}\right)+$$

$$+\eta_2\left[\frac{\partial}{\partial z}\frac{(\partial v_\theta/\partial z)}{I^{\frac{1}{2}}}+\frac{\partial}{\partial r}\frac{(\partial v_\theta/\partial r)-(v_\theta/r)}{I^{\frac{1}{2}}}+\frac{2}{r}\frac{(\partial v_\theta/\partial r)-(v\ /r)}{I^{\frac{1}{2}}}\right].$$

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta_1\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{\partial^2 v_z}{\partial z^2}\right)+$$

$$+\eta_2\left[\frac{\partial}{\partial z}\frac{2(\partial v_z/\partial z)}{I^{\frac{1}{2}}}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{(\partial v_r/\partial z)+(\partial v_z/\partial r)}{I^{\frac{1}{2}}}\right] \quad (1)$$

where,

$$I=2\left(\frac{\partial v_r}{\partial r}\right)^2+2\left(\frac{v_r}{r}\right)^2+2\left(\frac{\partial v_z}{\partial z}\right)^2+\left(\frac{\partial v_\theta}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}\right)^2+\left(\frac{\partial v_z}{\partial r}\right)^2+$$

$$+\left(\frac{\partial v_\theta}{\partial r}\right)^2+2\frac{\partial v_r}{\partial z}\frac{\partial v_z}{\partial r}-2\frac{\partial v_\theta}{\partial r}\frac{v_\theta}{r}+\left(\frac{v_\theta}{r}\right)^2$$

Equation of continuity is

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0 \quad (2)$$

If d be the characteristic radius of a disc and ω be its characteristic velocity, then it is assumed that

$$\frac{h}{d}\ll 1\ ,\quad \frac{\rho\ \omega\ h^2}{\eta_1}<1 \quad (3)$$

where ω is the greater of the two velocities, ω_1, the angular velocity of the inner disc and ω_2, the angular velocity of the outer disc. In most of the practical cases the flow is mainly due to relative rotation and very small radial flow due to the feeding of the lubricant.

We suppose that the motion of the fluid is mainly rotational, so that if $v^0{}_r$ is the characteristic radial velocity, then

$$v^0{}_r\ll\omega r$$

and hence the order of $v^0{}_r$ is assumed to be given by

$$v^0{}_r\sim\omega d\ \frac{\omega\rho h^2}{\eta_1} \quad (4)$$

If $v^0{}_z$ is characteristic axial velocity, then from the equation of continuity (2) it follows that

$$v^0{}_z\sim\frac{h}{d}\,v^0{}_r$$

Making use of the assumptions (3) and (4) and neglecting terms of the lower order, (1) is reduced to

$$-\frac{v^2_\theta}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\eta_1}{\rho}\frac{\partial}{\partial z}\left[\frac{\partial v_r}{\partial z}\left(1+\frac{\eta_2}{\eta_1 I^{\frac{1}{2}}}\right)\right]\ \frac{\partial}{\partial z}\left[\frac{\partial v_\theta}{\partial z}\left(1+\frac{\eta_2}{\eta_1 I_1^{\frac{1}{2}}}\right)\right]=0 \quad (5)$$

$$\frac{\partial p}{\partial z} = 0$$

where,

$$I_1 = \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2$$

BOUNDARY CONDITIONS

The system of equations (5) together with the continuity equation (2) is solved under the following boundary conditions:

$$\left.\begin{array}{lll} v_r = v_z = 0, \quad v_\theta = \omega_1 r & \text{where} & z = 0 \text{ and } R_1 \leqslant r \leqslant R_2 \\ v_r = v_z = 0, \quad v_\theta = \omega_2 r & \text{where} & z = h \text{ and } R_1 \leqslant r \leqslant R_2 \\ p = p_1 & \text{where} & r = R_1 \text{ and } 0 \leqslant z \leqslant h \\ p = p_2 & \text{where} & r = R_2 \text{ and } 0 \leqslant z \leqslant h \end{array}\right\} \quad (6)$$

SOLUTION OF THE PROBLEM

Dimensionless quantities are defined by a prime as follows:

$$v_\theta = v_\theta' \ \omega d, \quad v_r = v_r' \, v_r^0,$$

$$z = z' h, \quad p = \rho \ \omega^2 d^2 \ p', \quad v_z = \frac{h}{d} \ v_r^0 \ v_z'$$

Let $B = \eta_2 \, h/\eta_1 \, \omega d, \quad v_r^0/\omega d = \epsilon, \quad \omega \rho h^2/\eta_1 = \alpha$

where B is Bingham number.

Applying these transformations we get

$$(I_1)^{\frac{1}{2}} = (\partial v_\theta / \partial z)^2 + (\partial v_r/\partial z)^2$$

$$= \frac{\omega d}{h} \left| (\partial v_\theta' / \partial z') \right| (1 + \epsilon^2 \phi^2)^{\frac{1}{2}} \quad (7)$$

where, $\phi = (\partial v_r'/\partial z')/(\partial v_\theta' / \partial z')$.

Equations (5) becomes

$$\frac{\partial v_\theta'}{\partial z'} = c_1 (r') - B \frac{\partial v_\theta'/\partial z'}{|\partial v_\theta'/\partial z'|} \Big/ (1 + \epsilon^2\phi^2)^{\frac{1}{2}} \quad (8)$$

$$\frac{\partial \phi}{\partial z'} = \frac{\alpha}{\epsilon c_1} \left(\frac{dp'}{dr'} - \frac{(v_\theta')^2}{r'}\right)$$

$$\frac{dp'}{dz'} = 0$$

Equation of continuity is transformed into

$$\frac{\partial v_z'}{\partial z'} + \frac{1}{r'} \frac{\partial}{\partial r'} (r' v_r') = 0 \quad (9)$$

As a first approximation it is assumed that $\epsilon \ll 1$, i.e., radial velocity is very much less than the angular velocity.

$$\frac{\partial v_\theta'}{\partial z'} = c_1 - B \frac{\partial v_\theta' / \partial z'}{|\partial v_\theta' / \partial z'|} = c_2 (r') \quad (10)$$

Integrating and using the boundary conditions from (6) we have

$$v_\theta' = [(\omega_2 - \omega_1) z' + \omega_1] \ r'/\omega \quad (11)$$

Thus, equations (8) give

$$\frac{\partial \phi}{\partial z'} = \frac{A}{[r' + (B\omega/\omega_1 - \omega_2)]}\left[\frac{dp'}{dr'} - \frac{r'}{2}(\omega_2 - \omega_1)^2 z'^2 + \omega_1^2 + 2\omega_1 z' (\omega_2 - \omega_1)\right]$$

where, $A = \alpha\omega/\epsilon(\omega_2 - \omega_1)$

Therefore,

$$\phi = \frac{A}{[r' + B\omega/(\omega_2 - \omega_1)]}\left[\frac{dp'}{dr'} z' - \frac{r'}{\omega^2}\left\{(\omega_2 - \omega_1)^2 \frac{z'^3}{3} + \right.\right.$$

$$\left.\left. + \omega_1^2 z' + 2\omega_1(\omega_2 - \omega_1)\frac{z'^2}{2}\right\}\right] + K_1$$

Substituting the value of v_θ' and ϕ in (7), we obtain

$$\frac{\partial v'_r}{\partial z'} = \frac{(\omega_2 - \omega_1) r'}{\omega}\left[\frac{A}{(r' + B\omega/|\omega_2 - \omega_1|)}\frac{dp'}{dr'} z' - \frac{r'}{\omega^2}\cdot\right.$$

$$\left.\cdot(\omega_2 - \omega_1)^2 \frac{z'^3}{3} + \omega_1^2 z' + 2\omega_1(\omega_2 - \omega_1)\frac{z'^2}{2} + K_1\right]$$

Integrating and using the boundary conditions from (6), we get

$$v'_r = \frac{\alpha r'}{\epsilon(r' + B\omega/|\omega_2 - \omega_1|)}\left[\frac{1}{2}\frac{dp'}{dr'}(z' - 1)z' - \frac{r'}{\omega^2}\cdot\right.$$

$$\left.\cdot(\omega_2 - \omega_1)^2 z'\left(\frac{z'^3 - 1}{12}\right) + \frac{\omega_1(\omega_2 - \omega_1)}{3} z'(z'^2 - 1) + \frac{\omega_1^2}{2} z'(z' - 1)\right] \quad (12)$$

Substituting the value of v'_r in the equation of continuity (9) and applying the boundary conditions on v'_z from (6), we have

$$\frac{d}{dr'}\frac{\alpha r'^2}{\epsilon(r' + |B\omega/(\omega_2 - \omega_1)|}\ \frac{1}{2}\frac{dp'}{dr'} - \frac{r'}{2\omega^2}\left(\frac{(\omega_2 - \omega_1)^2}{20} + \frac{\omega_1\omega_2}{6}\right) = 0$$

which on integration gives,

$$\frac{dp'}{dr'} = \frac{K_3}{\alpha r'^2}\epsilon\left(r' + \frac{B\omega}{\omega_2 - \omega_1}\right) + \frac{6r'}{\omega^2}\left[\frac{(\omega_2 - \omega_1)^2}{20} + \frac{\omega_1\omega_2}{6}\right]$$

Solving this and using the boundary conditions on pressure (from (6)) we get,

$$p = p_2 - \gamma\left(1 - \frac{r^2}{R_2^2}\right) - \frac{\left(p_1 - p_2 + \gamma\frac{X - 1}{X}\right)\left(\log\frac{R_2}{r} + B\omega\left(\frac{1}{r} - \frac{1}{R_2}\right)\Big/(\omega_2 - \omega_1)\right)}{\log\left(\frac{R_1}{R_2}\right) + \frac{B\omega d}{\omega_2 - \omega_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)} \quad (13)$$

where, $X = (R_2/R_1)^2$

$$\gamma = R_2^2\left[\frac{3}{20}(\omega_2 - \omega_1)^2 + \frac{1}{2}\omega_1\omega_2\right]$$

LIFTING FORCE OF STEP BEARING

Lifting force, when the feeding is done from the centre of the disc to the periphery, is given by

$$P_1 = 2\pi \int_{R_1}^{R_2} r\,(p - p_2)\,dr\,, \qquad R_1 \leqslant r \leqslant R_2$$

Substituting the value of p, from (13),

$$P^I \equiv P_1/\pi R_2^2\gamma = \beta_1 f_1 - f_2$$

where,

$$\beta_1 = (p_1 - p_2)/\gamma > 0$$

$$f_1 = \frac{\frac{1}{X}\left[(X-1)/\log X - 1 \right] + \left[2B\omega d\,(1 + 1/X - 2/X^{\frac{1}{2}})/R_2 \,|\, \omega_2 - \omega_1 \,|\, \log X \right]}{1 + [\,2B\omega d\,(X^{\frac{1}{2}} - 1)/R_2 \,|\, \omega_2 - \omega_1 \,|\, \log X\,]}$$

$$f_2 = -f_1 \frac{X-1}{X} + \frac{1}{2} + \frac{1}{2X^2} - \frac{1}{X}\,.$$

If the feeding of the lubricant is done from the periphery to the centre of the disc, then the lifting force is given by

$$P_2 = 2\pi \int_{R_1}^{R_2} r\,(p - p_1)\,dr \text{ in the region } R_1 \leqslant r \leqslant R_2$$

Again substituting the value of p from (13),

$$P^{II} \equiv P_2/\pi\, R_2^2\, \gamma = \beta_2\, f_3 - f_2$$

where,

$$\beta_2 = p_2 - p_1/\gamma > 0$$

$$f_3 = (1 - 1/X) - f_1$$

MOMENT OF FRICTION

Moment of friction acting on the discs is given by

$$M = 2\pi \int_{R_1}^{R_2} \left(\eta_1 + \frac{\eta_2}{(I)^{\frac{1}{2}}} \right) \left| \frac{\partial v_\theta}{\partial z} \right| r^2\, dr$$

Evaluating and putting it in the non-dimensional form,

$$\frac{Mh}{\pi\, \eta_1\,(\omega_2 - \omega_1) R_2^4} = \frac{1}{2}\left(1 - \frac{1}{X^2} \right) + \frac{2}{3} B \left(1 - \frac{1}{X^3} \right)$$

CRITERION FOR FLOW

The flow will take place throughout the region of the film if

$$I \geqslant \eta_2^2 \qquad (R_1 \geqslant r \leqslant R_2)$$

Hence from (7) the condition for no-core formation is

$$\frac{\omega^2 d^2}{h^2} \left(\frac{\partial v_\theta'}{\partial z'} \right)^2 (1 + \epsilon^2 \phi^2) \geqslant \eta_2^2 \quad (R_1 \leqslant r \leqslant R_2)$$

For the case $\epsilon \ll 1$; from (10) and (11) the condition becomes

$$\frac{d^2}{h^2}\,(\omega_2 - \omega_1)^2\, r^2 \geqslant \eta_2^2\,.$$

As an example the yield value for the Grease sample with 10% soap content was found[3] to be 370 dynes/cm². The flow condition for this value of η_2 becomes $R_1 < R_2 > 370/(\omega_2 - \omega_1)^2$. Hence the bearing may be designed by choosing the values of R_1, R_2, ω_1 and ω_2 so as to satisfy this condition of flow.

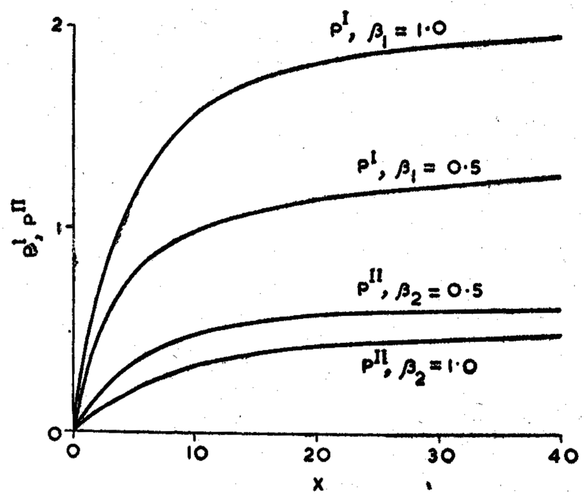

Fig. 2—Lifting force versus X (see Table for Fig. 2).

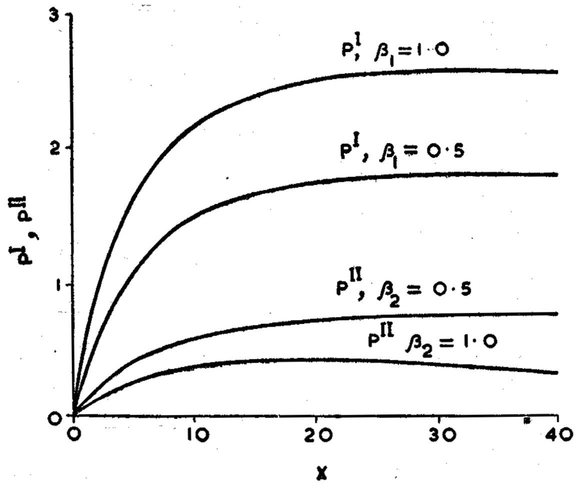

Fig. 3—Lifting force versus X (see Table for Fig. 3).

TABLE FOR FIG. 2

VALUES OF P^I AND P^{II} FOR DIFFERENT VALUES OF X WHEN $2B\,d/R_2\,(\omega_2 - \omega_1) = 1$

X	P^I		P^{II}	
	$\beta_1=0\cdot5$	$\beta_1=1\cdot0$	$\beta_2=0\cdot5$	$\beta_2=1\cdot0$
0	0	0	0	0
1	0	0	0	0
2	0·505	0·820	0	0
3	0·738	1·291	0·262	0·142
4	1·009	1·524	0·352	0·211
5	1·155	1·725	0·421	0·253
6	1·183	1·858	0·480	0·287
7	1·355	2·003	0·[illegible]06	0·304
8	1·417	2·068	0·545	0·329
9	1·473	2·148	0·563	0·342
10	1·515	2·314	0·594	0·358
20	1·789	2·532	0·695	0·425
30	1·873	2·683	0·761	—
40	1·905	2·724	0·781	—

TABLE FOR FIG. 3

VALUES OF P^I AND P^{II} FOR DIFFERENT VALUES OF X WHEN $2Bd/R_2\,(\omega_2 - \omega_1) = 2$

X	P^I		P^{II}	
	$\beta_1=0\cdot5$	$\beta_1=1\cdot0$	$\beta_2=0\cdot5$	$\beta_2=1\cdot0$
0	0	0	0	0
1	0	0	0	0
2	0·390	0·560	0	0
3	0·633	0·998	0·234	0·199
4	0·739	1·139	0·294	0·269
5	0·798	1·245	0·340	0·301

(contd.)

Table for Fig 3

6	0·863	1·301	0·372	0·329
7	0·885	1·352	0·415	0·351
8	0·948	1·426	0·417	0·367
9	0·969	1·459	0·432	0·378
10	0·994	1·484	0·444	0·396
20	0·999	1·655	0·492	0·415
30	1·103	1·728	0·526	0·465
40	1·104	1·785	0·563	0·472

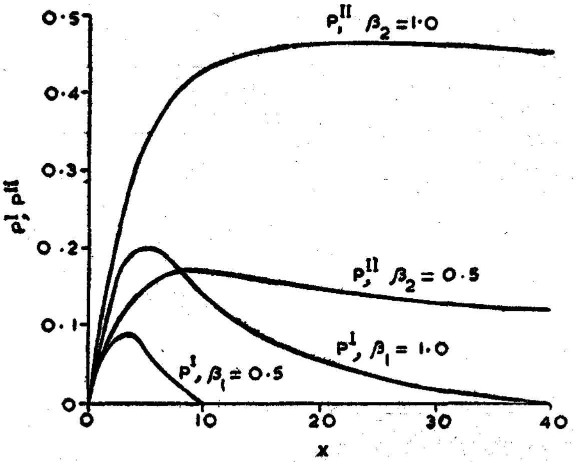

Fig. 4—Lifting force versus X (see Table for Fig. 4).

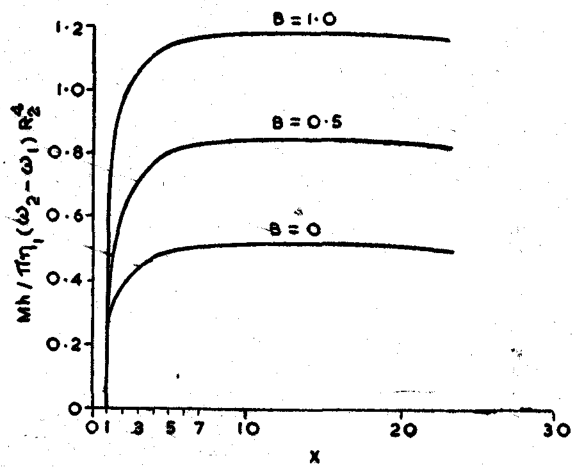

Fig. 5—Moment of friction versus X (see Table for Fig. 5).

TABLE FOR FIG. 4

LIFTING FORCES P^{I} AND P^{II} WHEN $B=0$

X	P^{I}		P^{II}	
	$\beta_1=0·5$	$\beta_1=1·0$	$\beta_2=0·5$	$\beta_2=1·0$
0	0	0	0	0
1	0	0	0	0
2	0·096	0·2071	0	0
3	0·097	0·233	0·157	0·353
4	0·083	0·228	0·166	0·396
5	0·066	0·215	0·169	0·421
6	0·051	0·199	0·175	0·433
7	0·036	0·185	0·166	0·446
8	0·025	0·172	0·165	0·455
9	0·017	0·163	0·163	0·462
10	0·002	0·147	0·161	0·488
20	—	0·069	0·139	—
30	—	0·025	0·133	—
40	—	0·003	0·128	—

TABLE FOR FIG. 5

MOMENT OF FRICTION (NON-DIMENSIONAL FORM) FOR VARIOUS VALUES OF X KEEPING B CONSTANT

X	1	2	3	4	5	10	20	30	40	50
$B=0$	0	0·3750	0·4440	0·4680	0·4800	0·4950	0·4980	0·4994	0·4997	0·4998
$B=0·5$	0	0·6666	0·7653	0·7968	0·8106	0·8280	0·8320	0·8327	0·8329	0·8331
$B=1·0$	0	0·9583	1·0863	1·1250	1·1413	1·1610	1·1653	1·1660	1·1663	1·1664

DISCUSSION AND CONCLUSION

The lifting force of the step bearing for the values of the Bingham number given by

$$\frac{2Bd}{R_2(\omega_2-\omega_1)} = 1 \quad \text{and} \quad \frac{2Bd}{R_2(\omega_2-\omega_1)} = 2$$

and for different values of $X = (R_2/R_1)^2$, has been calculated for two cases (a) when the feeding is done from the centre of the disc to the periphery, (b) when the feeding is done from the periphery to the centre of the disc. Tables of representative values and graphs are shown in the Fig 2 and 3. When Bingham number B is equal to zero, the Bingham solid behaves as a Newtonian lubricant having the viscosity value equal to the reciprocal mobility of the Bingham material. The values of the lifting force[2] in this case are shown in Fig. 4.

Moment of friction acting on the discs has been calculated for Bingham number $B = 0, 0\cdot5$ and $1\cdot0$ and for various values of $X = (R_2/R_1)^2$. Tables of representative values and graphs are demonstrated in the Fig 5. It is found that the value of Moment of friction is greater for a Newtonian lubricant having the same viscosity value as the reciprocal mobility of the Bingham material. Also it is noted that the moment of friction increases with the increase of Bingham number within the considered range.

Grease lubricated bearings are extensively used in the defence equipment. The findings in the present paper are useful for the design of bearings lubricated by grease or other Bingham lubricants.

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REFERENCES

1. OLDROYD, J. G., *Proc. Camp Phil. Soc.*, **43** (1947), 100.
2. TARAPOV, I. E., *Proc. USSR Academy of Sci.*, **2** (1959), 194.
3. BLOTT, J. F. T. & BONNOR, W. B., *Proc. 1st Rheology congress*, 1948.