GENERALISED THERMO-ELASTICITY WHEN THE MATERIAL COUPLING PARAMETER EQUALS UNITY

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Analytical solutions of three problems using the theory of generalised thermo-elasticity are presented for the case when the material coupling parameter equals unity ($\delta=1$). The problems considered are (1) Constant velocity impact, (2) Daniloviskaya's problem, and (3) Step in strain. Solutions are presented for the case of thin bars (One dimensional stress) and are obtained using Laplace transform. There is a great simplification in the equations of generalised thermo-elasticity when the material coupling parameter equals unity, which permits the straight forward inversion of the transformed solutions. The solutions obtained are more general which includes the effect of relaxation time also. The important feature of this paper is that the solutions of coupled theory can be readily obtained simply by putting the relaxation constant equals to zero ($\beta=0$).

Among the most important technological advances of the last three decades have been the development of nuclear sources of energy and the attainment of rocket-powered high speed flight. Severe stresses may be developed in a structure subjected to non-uniform changes in temperature. A knowledge of the thermal stresses is of great technical importance in the safe and economical design of aircraft structures. The latest advancement is the formulation of the Generalised Dynamical theory of thermo-elasticity.

In order to avoid paradox of an infinite velocity of propagation, inherent in the existing coupled theory of thermo-elasticity, Lord & Shulman[1] generalised the dynamical theory of thermo-elasticity which takes into account the effect of relaxation time i.e., the time needed for the onset of thermal wave. Their analysis results in a system of coupled hyperbolic equations. Dillon[2] solved the same problems (1), (2) and (3) in coupled theory and obtained solutions when the material coupling parameter δ equals unity. There he had given justification for this value of δ. He pointed out that there may appear situations where the linear approximation to a nonlinear material involves this value δ.

The purpose of this investigation is to study the three problems (1) Constant velocity impact of identical specimens, (2) Danilovskaya's problem of step function in temperature, (3) A step function in strain as described in[3], within the frame work of the theory developed[1], when the material coupling parameter equals unity.

FORMULATION AND SOLUTION OF THE PROBLEM

Consider a long thin rod in which the only non zero stress component is the axial one σ_{11}. The linearised equation of motion is

$$\frac{\partial \sigma_{11}}{\partial x_1} = \rho_0 \frac{\partial^2 u_1}{\partial t^2} \tag{1}$$

where u_1 is the displacement in the axial direction x_1, ρ_0 is the undeformed density and t is time.

The energy equation of the isotropic elasticity[1] is given by

$$K \frac{\partial^2\theta}{\partial x_1^2} = \rho_0 C_D \left(\frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2\theta}{\partial t^2}\right) + (3\lambda + 2\mu)\, \alpha\, T_0\left(\frac{\partial e}{\partial t} + \tau_0 \frac{\partial^2 e}{\partial t^2}\right) \tag{2}$$

where θ is the temperature increment above the uniform temperature T_0. K, C_D, α and τ_0 are respectively the thermal conductivity, specific heat, coefficient of linear thermal expansion and relaxation time. λ and μ are Lami's constants. Equation (2) also assumes the axial flow of heat in order to be consistent with the bar approximation. The constitutive equation for the isotropic linear elastic solid is

$$(3\lambda + 2\mu)\, e = \sigma_{11} + 3\alpha\, (3\lambda + 2\mu)\, \theta \tag{3}$$

For the case of thin rod

$$E\epsilon_{11} = \sigma_{11} + \alpha E\theta \tag{4}$$

Where E is Young's modulus.

Substitution of equation (3) into (2) yields

$$K \frac{\partial^2\theta}{\partial x^2_1} = \rho_0 C\sigma \frac{\partial\theta}{\partial t} + \rho_0 C\sigma \frac{\partial^2\theta}{\partial t^2} + \alpha T_0 \left(\frac{\partial\sigma_{11}}{\partial t} + \tau_0 \frac{\partial^2\sigma_{11}}{\partial t^2}\right) \tag{5}$$

where

$$C\sigma = C_D + \frac{3\alpha^2 T_0 (3\lambda + 2\mu)}{\rho_0}.$$

Equation (5) is more appropriate than equation (2), because only one stress component exits whereas three strain components are permitted. The strain ϵ_{11} is related to the axial displacement by the definition

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \tag{6}$$

It is convenient to consider the above equations for the displacements, strain or temperature which do not involve the other variables. It is also convenient to have these expressions in non-dimensional form.

Now introducing the following non-dimensional variables

$$x = x_1/a\,,\ \tau = Vt/a,\ u = u_1/a$$

where

$$a = K/\rho_0 C\sigma V \quad \text{and} \quad V = (E/\rho_0)^{1/2}$$

$$\text{Coupling parameter } (\delta) = \frac{\alpha^2 T_0 E}{\rho_0 C\sigma} \tag{7}$$

$$\text{Relaxation constant } (\beta) = \frac{\rho_0 C\sigma \tau_0 V^2}{K}$$

The above field equation reduces to the form

$$u'''' - (1+\beta)\, u^{\cdot\cdot}{}'' - u^{\cdot}{}'' + (1-\beta)\, u^{\cdot\cdot\cdot} + \beta(1-\delta)\, u^{\cdot\cdot\cdot\cdot} = 0 \tag{8}$$

Where primes denote differentiation with respect to x and superposed dots denote differentiation with respect to time τ.

In a similar manner, we can deduce the stress equation and temperature equation in the form

$$\sigma_{11}'''' - (1+\beta)\,\sigma_{11}{}^{\cdot\cdot}{}'' - \sigma_{11}{}^{\cdot}{}'' + (1-\delta)\,\sigma_{11}{}^{\cdot\cdot\cdot} + \beta(1-\delta)\sigma_{11}{}^{\cdot\cdot\cdot\cdot} = 0 \tag{9}$$

and

$$\theta'''' - (1+\beta)\,\theta^{\cdot\cdot}{}'' - \theta^{\cdot}{}'' + (1-\delta)\,\theta^{\cdot\cdot\cdot} + \beta(1-\delta)\,\theta^{\cdot\cdot\cdot\cdot} = 0 \tag{10}$$

which of these equations convenient depends on the boundary conditions of the particular problem.

The following equations can be obtained from the foregoing basic equations.

$$\alpha\,\theta' = u'' - u^{\cdot\cdot} \tag{11}$$

$$\alpha E\,\theta^{\cdot\cdot} = \sigma_{11}'' - \sigma_{11}{}^{\cdot\cdot} \tag{12}$$

$$\alpha E\,(\theta'' - \theta^{\cdot} - \beta\,\theta^{\cdot\cdot}) = \delta\,\sigma_{11}{}^{\cdot} + \beta\delta\sigma_{11}{}^{\cdot\cdot} \tag{13}$$

$$\alpha\theta^{\cdot}\,(1-\delta) + \beta(1-\delta)\,\alpha\theta^{\cdot\cdot} = u''' - (1+\delta\beta)\,u^{\cdot\cdot}{}' - \delta u^{\cdot}{}' \tag{14}$$

In order to find the solutions of the problems, Laplace transform technique is used. Let capital letters denote the transform variables and p be the transform parameter, thus

$$U(x,p) = \int_0^\infty e^{-p\tau}\, u(x,\tau)\, d\tau. \tag{15}$$

Similarly $\Sigma\,(x,p)$ and $\theta\,(x,p)$ are the Laplace transforms of $\sigma_{11}\,(x,\tau)$ and $\theta\,(x,\tau)$.

Now we will consider the three problems.

CONSTANT VELOCITY IMPACT

Constant velocity impact is equivalent to the motion of the one end of the specimen moving at a prescribed speed V_0. The corresponding mechanical boundary conditions are

$$u(0,\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ V_0\,\tau & \text{for } \tau > 0 \end{cases} \tag{16}$$

and when symmetrical specimens are used, no heat flows across the surface. i.e.,

$$\theta'(0,\tau) = 0 \tag{17}$$

The initial conditions are assumed to be

$$u(x,0) = u^{\cdot}(x,0) = u^{\cdot\cdot}(x,0) = u^{\cdot\cdot\cdot}(x,0) = \theta(x,0) = 0$$

and the usual regularity conditions as x approaches infinity are used.

Applying Laplace transform and proceeding in the same way as in Dillon's[2], we get

$$U = \frac{V_0(\lambda_2^2 - p^2)}{p^2(\lambda_2^2 - \lambda_1^2)}\, e^{-\lambda_1 x} - \frac{V_0(\lambda_1^2 - p^2)}{p^2(\lambda_2^2 - \lambda_1^2)}\, e^{-\lambda_2 x} \tag{18}$$

when $\delta = 1$, λ_1 and λ_2 are given by

$$\lambda_1 = [p((1+\beta)p + 1)]^{1/2} \tag{19}$$
$$\lambda_2 = 0$$

The strain is given by

$$U' = -\frac{V_0}{[(1+\beta)p^2 + p]^{1/2}}\, e^{-x[(1+\beta)p^2 + p]^{1/2}} \tag{20}$$

The inverse transform of (20) is well known[4] and is

$$\left.\begin{aligned} u'(x,\tau) &= 0 \quad \text{when } 0 < \tau < \sqrt{(1+\beta)}\,x \\ &= -\frac{V_0}{\sqrt{1+\beta}}\, e^{-[\tau/2(1+\beta)]}\, T_0\left[\sqrt{\tau^2 - (1+\beta)x^2}/2(1+\beta)\right] \end{aligned}\right\} \tag{21}$$

$$\text{when } \tau > \sqrt{(1+\beta)}\,x$$

Where I_0 is the modified Bessel function of order zero.

The jump in strain at the wave front is given by

$$-\frac{V_0}{\sqrt{1+\beta}}\, e^{-[x/2\sqrt{1+\beta}]} \tag{22}$$

The temperature becomes

$$\begin{aligned} \alpha\theta = &-\frac{V_0 e^{-\left[\sqrt{(1+\beta)}x\left(p^2 + \frac{p}{1+\beta}\right)^{1/2}\right]}}{(1+\beta)[p + 1/(1+\beta)]\,[p^2 + p/(1+\beta)]^{1/2}\sqrt{1+\beta}} \\ &- \frac{V_0\beta p\, e^{-[\sqrt{(1+\beta)}x(p^2 + p/(1+\beta))^{1/2}]}}{(1+\beta)[(1+\beta)p + 1]\,[p^2 + p/(1+\beta)]^{1/2}} \\ &- \frac{V_0 x}{(1+\beta)[p + 1/(1+\beta)]} - \frac{V_0\beta p x}{(1+\beta)[p^2 + 1/(1+\beta)]} \end{aligned} \tag{23}$$

Taking inverse,

$$\frac{a\theta}{V_0} = \frac{e^{-\tau/(1+\beta)}}{(1+\beta)} \left[x - \frac{1}{\sqrt{1+\beta}} \int_{\sqrt{1+\beta}\,x}^{\tau} e^{\frac{\eta}{2(1+\beta)}} \cdot I_0 \left\{ \frac{1}{2(1+\beta)} \sqrt{\eta^2-(1+\beta)\,x^2} \right\} d\eta \right] + \frac{\beta}{(1+\beta)^2}\, e^{-\tau/(1+\beta)}$$

$$\left[x + \frac{1}{\sqrt{1+\beta}} \int_{\sqrt{(1+\beta)}\,x}^{\tau} e^{\frac{\eta}{2(1+\beta)}} I_0 \left\{ \frac{1}{2(1+\beta)} \sqrt{\eta^2-(1+\beta)\,x^2} \right\} d\eta \right]$$

when $\tau > \sqrt{(1+\beta)}\,x$

$$= \frac{x}{(1+\beta)}\, e^{-\frac{\tau}{(1+\beta)}} + \frac{\beta x}{(1+\beta)^2}\, e^{-\tau/(1+\beta)}$$

when $\tau < \sqrt{(1+\beta)\,x}$ (24)

The stress is obtained from (4) as

$$\sigma_{11}(x,\tau) = E\,V_0\, e^{-\tau/(1+\beta)} \left[-\frac{x}{(1+\beta)} - \frac{\beta x}{(1+\beta)^2} - \right.$$

$$- \frac{e^{\tau/2(1+\beta)}}{\sqrt{1+\beta}}\, I_0 \left\{ \frac{1}{2(1+\beta)} \sqrt{\eta^2-(1+\beta)\,x^2} \right\} +$$

$$+ \frac{1}{\sqrt{1+\beta}} \left\{ \frac{1}{(1+\beta)} - \frac{\beta}{(1+\beta)^2} \right\} \int_{\sqrt{1+\beta\,x}}^{\tau} e^{\eta/2(1+\beta)}$$

$$\left. \cdot\, I_0 \left\{ \frac{1}{2(1+\beta)} \sqrt{\eta^2-(1+\beta)\,x^2} \right\} \right] d\eta$$

when $\tau > \sqrt{1+\beta\,x}$

$$= E\, V_0\, e^{-\tau/(1+\beta)} \left[-\frac{x}{(1+\beta)} - \frac{\beta x}{(1+\beta)^2} \right]$$

when $\tau < \sqrt{(1+\beta)}x$ (25)

DANILOVSKAYA'S PROBLEM

The problem considered is that of a step function in prescribed temperature on the surface $x = 0$ and no stress is applied on this face. There are no infinite stress, displacements, or temperatures in this problem, and the infinite bar is completely permissible. The boundary conditions are

$$\sigma_{11}(0,\tau) = 0$$
$$\theta(0,\tau) = 0, \qquad \tau < 0$$
$$T_1, \qquad \tau > 0 \tag{26}$$

and the usual regularity conditions at $x = \infty$ and quiescent initial conditions are assumed.

Using the Laplace transform and proceeding in the same way as in[2], we get

$$\theta = \frac{T_1(1+\beta p)}{p\,[1+(1+\beta)\,p]}\, e^{-[(1+\beta)p^2+p]^{1/2} x} + \frac{T_1}{[1+(1+\beta)\,p]} \tag{27}$$

Inverting above equation by using tabulated pairs[5], we get

$$\theta(x,\tau) = T_1 \left[\int_{\sqrt{(1+\beta)}\,x}^{\tau} \left[1 - e^{-\frac{(\tau-\eta)}{(1+\beta)}} \right] \left[e^{-x/2(\sqrt{1+\beta})} + \frac{2x}{\sqrt{1+\beta}} e^{-\eta/2(1+\beta)} \right.\right.$$

$$\left.\left. \cdot \frac{I_1\left[\frac{2}{(1+\beta)}\sqrt{\eta^2-(1+\beta)\,x^2}\right]}{\sqrt{\eta^2-(1+\beta)\,x^2}} \right] d\eta \right] +$$

$$+ \frac{\beta T_1}{(1+\beta)} \int_{\sqrt{(1+\beta)}\,x}^{\tau} e^{-(\tau-\eta)/(1+\beta)} \left[e^{-x/2\sqrt{(1+\beta)}} + \frac{2x\, e^{-\eta/2\sqrt{1+\beta}}}{\sqrt{1+\beta}} \right.$$

$$\left. \cdot \frac{I_1\left[\frac{2}{(1+\beta)}\sqrt{\eta^2-(1+\beta)\,x^2}\right]}{\sqrt{\eta^2-(1+\beta)x^2}} \right] d\eta +$$

$$+ \frac{T_1}{(1+\beta)} e^{-\tau/(1+\beta)} \qquad \text{when } \tau > \sqrt{(1+\beta)}\ x$$

$$= T_1 \left[\int_{\sqrt{(1+\beta)}\,x}^{\tau} \left[1 - e^{-(\tau-\eta)/(1+\beta)} \right] \left[\frac{2x}{\sqrt{(1+\beta)}} e^{-\eta/2(1+\beta)} \right.\right.$$

$$\left.\left. \frac{I_1\left[\frac{2}{1+\beta}\sqrt{\eta^2-(+\beta)\,x^2}\right]}{\sqrt{\eta^2-(1+\beta)\,x^2}} \right] d\eta \right] +$$

$$+ \frac{\beta T_1}{(1+\beta)} \int_{\sqrt{(1+\beta)}\,x}^{\tau} e^{-(\tau-\eta)/(1+\beta)} \left[\frac{2x}{\sqrt{(1+\beta)}} e^{-\eta/2(1+\beta)} \right.$$

$$\left. \cdot \frac{I_1\left[\frac{2}{(1+\beta)}\sqrt{\eta^2-(1+\beta)\,x^2}\right]}{\sqrt{\eta^2-(1+\beta)\,x^2}} \right] d\eta + \frac{T_1}{(1+\beta)} e^{-\tau/(1+\beta)} \quad \text{when } \tau < \sqrt{(1+\beta)}x \qquad (28)$$

The stress produced in this problem is obtained from equations (13) and (27), as

$$\Sigma = \frac{\alpha E T_1}{[1+(1+\beta)p]} \left[1 - e^{-[(1+\beta)p^2+p]^{\frac{1}{2}}\ x} \right] \qquad (29)$$

Inverting, we get

$$\sigma_{11}(x,\tau) = \frac{\alpha E T_1}{1+\beta} \left[e^{-\tau/(1+\beta)} - \int_{\sqrt{(1+\beta)}\,x}^{\tau} e^{-(\tau-\eta)/(1+\beta)} \right.$$

$$\left\{ e^{-x/2\sqrt{1+\beta}} + \frac{2x}{\sqrt{1+\beta}} \, e^{-\eta/2(1+\beta)} \,\cdot\, \frac{I_1\left[\frac{2}{(1+\beta)}\sqrt{\eta^2-(1+\beta)x^2}\right]}{\sqrt{\eta_2-(1+\beta)x^2}} \right] d\eta \Bigg\}$$

$$\text{when } \tau > \sqrt{(1+\beta)}\, x \tag{30}$$

$$= \frac{\alpha E T_1}{(1+\beta)} \left[e^{-\tau/(1+\beta)} - \int_{\sqrt{(1+\beta)}\,x}^{\tau} e^{-(\tau-\eta)/(1+\beta)} \right.$$

$$\left[\frac{2x}{\sqrt{1+\beta}} \, e^{-\eta/2(1+\beta)} \, \frac{I_1\left[(2/1+\beta)\sqrt{\eta^2-(1+\beta)x^2}\right]}{\sqrt{\eta^2-(1+\beta)x^2}}\right] d\eta \Bigg]$$

$$\text{when } \tau < \sqrt{(1+\beta)}\, x$$

STEP IN STRAIN

This problem was solved[3] for the half space and for small δ. The boundary conditions are

$$u'(0,\tau) = \begin{cases} 0, & \tau < 0 \\ \epsilon_0, & \tau > 0 \end{cases} \tag{31}$$

and $\theta(0,\tau) = 0$.

and the regularity conditions at $x = \infty$.

Using the Laplace transform and proceeding in the same was as in[2], we get when $\delta = 1.0$, the strain is given by

$$U' = \frac{\epsilon_0}{p} \, e^{-[(1+\beta)p^2+p]^{1/2} x} \tag{32}$$

Inverting, we get

$$u'(x,\tau) = \int_{\sqrt{(1+\beta)}\,x}^{\tau} \left\{ e^{-x/2(\sqrt{1+\beta})} + \frac{2x}{\sqrt{1+\beta}} \, e^{-\eta/2(1+\beta)} \,\cdot\, \frac{I_1\left[\frac{2}{1+\beta}\sqrt{\eta^2-(1+\beta)x^2}\right]}{\sqrt{\eta^2-(1+\beta)x^2}} \right\} d\eta$$

$$\text{when } \tau > \sqrt{(1+\beta)}\, x$$

$$= \epsilon_0 \int_{\sqrt{(1+\beta)}\,x}^{\tau} \left\{ \frac{2x}{\sqrt{(1+\beta)}} \, e^{-\eta/2(1+\beta)} \, \frac{I_1\left[\frac{2}{1+\beta}\sqrt{\eta^2-(1+\beta)x^2}\right]}{\sqrt{\eta^2-(1+\beta)x^2}} \right\} d\eta.$$

$$\text{when } \tau < \sqrt{(1+\beta)}\, x. \tag{33}$$

The temperature is obtained from equation (11)

$$\alpha\theta' = U'' - p^2 U$$

$$\alpha\theta = \epsilon_0 \left[\frac{1}{p[(1+\beta)p+1]} + \frac{\beta}{p[(1+\beta)p+1]} \right] e^{-[(1+\beta)p^2+p]^{1/2} x} \tag{34}$$

Inverting we get

$$\alpha\theta = \epsilon_0 (1+\beta) \left[\int_{\sqrt{(1+\beta)}\, x}^{\tau} \left[1 - e^{-(\tau-\eta)/(1+\beta)} \right] \left[e^{-x/2(1+\beta)} + \right. \right.$$

$$\left. \left. + \frac{2x}{\sqrt{1+\beta}} e^{-\eta/2(1+\beta)} \frac{I_1\left[\frac{2}{1+\beta}\sqrt{\eta^2-(1+\beta)x^2}\right]}{\sqrt{\eta^2-(1+\beta)x^2}} \right] d\eta \right]$$

$$\text{when } \tau > \sqrt{(1+\beta)}\, x$$

$$= \epsilon_0 (1+\beta) \left[\int_{\sqrt{(1+\beta)}\, x}^{\tau} \left[1 - e^{-(\tau-\eta)/(1+\beta)} \right] \left[\frac{2x}{\sqrt{1+\beta}} e^{-\eta/2(1+\beta)} . \right. \right.$$

$$\left. \left. \cdot \frac{I_1\left[\frac{2}{1+\beta}\sqrt{\eta^2-(1+\beta)x^2}\right]}{\sqrt{\eta^2-(1+\beta)x^2}} \right] d\eta \right]$$

$$\text{when } \tau < \sqrt{(1+\beta)}\, x \qquad (35)$$

The stress distribution can be obtained by using equations (4), (33) and (35).

DISCUSSION AND RESULTS

The strain history of velocity impact problem for the value of $\delta=0$, $\beta=0.021$ and for $x=0.2$ and $x=0.8$ is shown in the Fig. 1 and is compared with the existing solution of Dillon[2]. From the Fig. 1, it is evident that the present solution deviates slightly from that of O.W. Dillon's, which shows the effect of

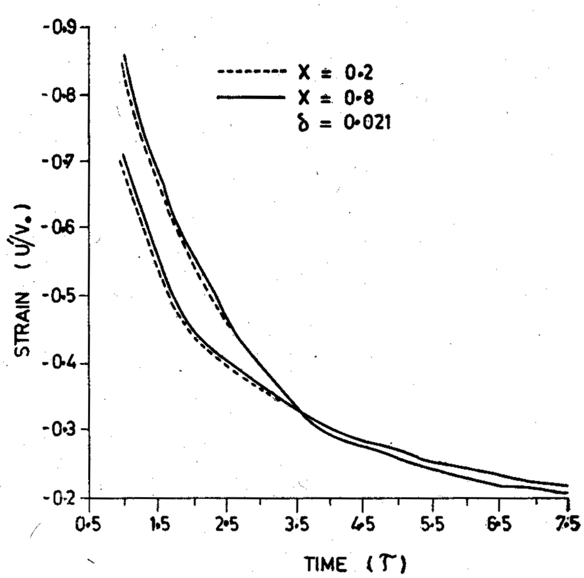

Fig. 1— Problem of velocity impact.

relaxation time. It is also observed that for large values of x and τ, the solution obtained in this problem is almost same as that of O.W. Dillon's, which shows that the effect of relaxation time will be felt only for small values of x and τ.

CONCLUSION

The solutions for the three problems : (1) Velocity impact (2) Danilovskaya's problem (3) Step in strain are obtained when the material coupling parameter equals unity ($\delta=1$), using the generalised theory of thermo-elasticity. The justification for this value of δ was given by Dillon[2]. The solutions obtained are more general one. The solutions of the same problems in coupled theory can be obtained as a particular case from the solutions by putting $\beta=0$ in equations (20), (23), (27), (29) and (32).

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