MINIMUM DRAG CONFIGURATIONS IN HYPERSONIC FLOW VIA THE METHOD OF STEEPEST DESCENT

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It has been shown that the problem of finding minimum drag configuration in hypersonic flow can be reduced to a problem in optimum control and the method of steepest ascent is a useful tool for finding the solution of this problem.

With the help of the indirect method of the calculus of variations there have been a few studies [1–3] of the problem of determining minimum drag configurations in hypersonic flow regime. Recently with the advent of computers and evaluation of modern theories of optimization there is an increasing trend towards direct methods and a number of systematic numerical procedures have been developed. One such approach which has proved to be a powerful numerical computing tool for the optimization of controlled plant when the quantity to be optimized is a function of the final values of the dependent variables is the method of steepest descent[4]. Though this method has found an extensive application in control theory its application in aerodynamics especially to problems of finding optimal shapes has not been seriously considered. The basic idea behind this method is to obtain equations for adjusting estimates of the decision functions in order to improve the value of the objective. In the present communication it has been brought out how the method of steepest descent can also be applied for finding minimum drag shapes in hypersonic flow. For the sake of illustrating the application of the steepest descent algorithm the simple case of slender body having minimum pressure drag has been considered and it has been established that more complicated problems can be solved utilising this method.

STEEPEST DESCENT METHOD AND THE ALGORITHM

The basic features of the steepest descent method and the steps required to solve a problem will now be described. It is required to find the decision function θ (t) so as to minimise

$$\phi = \phi\,[x_i\,(t_f)] \qquad i = 1, 2, \ldots\ldots, n \tag{1}$$

where

$$\dot{x}_i = f_i\,(x_i,\ \theta,\ t) \qquad x_i\,(t_0) \text{ is specified} \quad i = 1,\ 2, \ldots\ldots, n \tag{2}$$

$$\psi\,[x_i\,(t_f)] = 0 \qquad i = 1,\ 2, \ldots\ldots, n \tag{3}$$

where t_0 and t_f being initial and final instants of time and are specified.

The essential steps of the algorithm are as follows ;

I. Initially estimate θ (t) history,

II. Using this estimate integrate the system of equation (2) with known initial conditions,

III. Calculate and record $\phi(t_f)$ and $\psi(t_f)$,

IV. Find the influence functions λ^{ϕ} corresponding to ϕ and λ^{ψ} corresponding to ψ by backward integration of the influence functions equations,

$$\dot{\lambda}_i^{\phi} = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}\,\lambda_j^{\phi} \qquad \lambda_i^{\phi}\,(t_f) = \left.\frac{\partial \phi}{\partial x_i}\right|_{t = t_f} \quad i = 1,\ 2, \ldots\ldots, n$$

$$\dot{\lambda}_i^{\psi} = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}\,\lambda_j^{\psi} \qquad \lambda_i^{\psi}\,(t_f) = \left.\frac{\partial \psi}{\partial x_i}\right|_{t = t_f} \quad i = 1,\ 2, \ldots\ldots, n$$

V. Simultaneously with IV determine the following three integrals

$$I_{\phi\phi} = \int_0^1 \left(\sum_{i=1}^{n} \lambda_i^{\phi} \frac{\partial f_i}{\partial \theta} \right)^2 dt$$

$$I_{\phi\psi} = \int_0^1 \left(\sum_{i=1}^{n} \lambda_i^{\phi} \frac{\partial f_i}{\partial \theta} \right) \left(\sum_{i=1}^{n} \lambda_i^{\psi} \frac{\partial f_i}{\partial \theta} \right) dt$$

$$I_{\psi\psi} = \int_0^1 \left(\sum_{i=1}^{n} \lambda_i^{\psi} \frac{\partial f_i}{\partial \theta} \right)^2 dt$$

VI. Calculate the expression for the change in control function from the expression

$$\delta \theta (t) = K_\phi \left(\sum_{i=1}^{n} \lambda_i^{\phi} \frac{\partial f_i}{\partial \theta} \right) + K_\psi \left(\sum_{i=1}^{n} \lambda_i^{\psi} \frac{\partial f_i}{\partial \theta} \right)$$

where K_ϕ and K_ψ are constants given by

$$\delta\phi = K_\phi I_{\phi\phi} + K_\psi I_{\psi\phi}$$

$$\delta\psi = K_\phi I_{\psi\phi} + K_\psi I_{\psi\psi}$$

where $\delta\phi$ and $\delta\psi$ are asked for changes in ϕ and ψ respectively.

VII. Calculate the quantity $I_{\phi\phi} - I_{\phi\psi} I^{-1}_{\psi\psi} I_{\psi\phi}$

VIII. Stop if the quantity calculated in step VII tends towards zero, otherwise repeat the above with

$$\bar{\theta} (t) = \theta (t) + \delta\theta (t)$$

FORMULATION OF THE PROBLEM

We will now see how the problem of finding the minimum drag shapes can be reduced to the problem described above and how the steepest descent method can be applied for its solution. Considering the Newtonian flow theory and assuming that the slender body is at zero angle of attack, drag, volume and surface area of the body are given by

$$\frac{D}{4\pi q} = \int_0^l y\, y'^3\, dx \tag{4}$$

$$\frac{V}{\pi} = \int_0^l y^2\, dx \tag{5}$$

$$\frac{S}{2\pi} = \int_0^l y\, dx \tag{6}$$

where

$$y' = dy/dx$$

Introducing the dimensionless variables at $t = x/l$ and $X_3 = y/l$ and putting $y^1(x) = \theta$, the set of equations (4) to (6) can be written as

$$\frac{D}{4\pi q l^2} = \int_0^1 X_3\, \theta^3\, dt$$

$$\frac{V}{\pi l^3} = \int_0^1 X_3^2\, dt$$

$$\frac{S}{2\pi l^2} = \int_0^1 X_3\, dt$$

If we consider t as the independent variable and introduce the following definitions

$$X_1 = \frac{D}{4\pi q l^2}\,,\; X_2 = \frac{V}{\pi l^3}\,,\; X_4 = \frac{S}{2\pi l^2}$$

we arrive at

$$f_1 \equiv \dot{X}_1 = X_3\, \theta^3 \tag{7}$$

$$f_2 \equiv \dot{X}_2 = X_3^2 \tag{8}$$

$$f_3 \equiv \dot{X}_3 = \theta \tag{9}$$

$$f_4 \equiv \dot{X}_4 = X_3 \tag{10}$$

We will consider the following six cases and for minimum drag the quantity to be minimised $\phi(1)$ and the end condition $\psi(1)$ to be satisfied in each case are as follows :—

(i) Length and diameter of the body prescribed

$$\phi(1) \equiv X_1(1)\,;\; \psi(1) \equiv X_3(1) - \frac{\tau}{2} = 0\,,\; \tau = \frac{d}{l}$$

(ii) Diameter and surface area of the body prescribed

$$\phi(1) \equiv \frac{X_1(1)}{X_3^2(1)}\,;\; \psi(1) \equiv \frac{X_4(1)}{X_3^2(1)} - \alpha = 0\,,\; \alpha = \frac{2S}{\pi d^2}$$

(iii) Diameter and Volume of the body prescribed

$$\phi(1) \equiv \frac{X_1(1)}{X_3^2(1)}\,;\; \psi(1) \equiv \frac{X_2(1)}{X_3^2(1)} - \beta = 0\,,\; \beta = \frac{8V}{\pi d^3}$$

(iv) Length and surface area of the body prescribed

$$\phi(1) \equiv X_1(1)\,;\; \psi(1) \equiv X_4(1) - \gamma = 0\,,\; \gamma = \frac{S}{2\pi l^2}$$

(v) Length and volume of the body prescribed

$$\phi(1) \equiv X_1(1)\,;\; \psi(1) \equiv X_2(1) - \mu = 0\,,\; \mu = \frac{V}{\pi l^3}$$

(vi) Surface area and volume of the body prescribed

$$\phi(1) \equiv \frac{X_1(1)}{X_4(1)}\,;\; \psi(1) \equiv \frac{X_2^2(1)}{X_4^2(1)} - \nu = 0\,,\; \nu = \frac{8\pi V^2}{S^3}$$

Thus in the language of control theory our problem may be stated as follows :

Taking θ to be the decision variable and X_1, X_2, X_3, X_4 to be state variables, it is required to find the history of θ, i.e., $\theta = \theta\ (t)$ so that the value of ϕ (1) is minimum and the system of equations (7) to (10) together with the end condition ψ (1) are satisfied in each case.

SOLUTION OF THE PROBLEM

Now in what follows it is shown how the method of steepest descent can be applied to solve the problem in the cases mentioned above. The success of the algorithm described depends upon the proper intitial choice of the control variable history θ (t). In previous studies optimal shapes have been obtained via variational technique and we will here verify that these shapes actually satisfy the steepest descent algorithm and thus establish the usefulness of this powerful mathematical tool for solving optimal shape problems.

Case (1)—(l,d) *prescribed*

In this case the surface area and volume are free and the only prescribed quantities are length and diameter so the equations (8) and (10) can be ignored from the analysis. Thus in this case the system of equations are

$$\left.\begin{aligned} f_1 &\equiv \dot{X}_1 = X_3\,\theta^3 \\ f_3 &\equiv \dot{X}_3 = \theta \end{aligned}\right\} \quad X_1\,(0) = X_3\,(0) = 0 \tag{11}$$

with

$$\phi\,(1) \equiv X_1\,(1) \text{ and } \psi\,(1) \equiv X_3\,(1) - \frac{\tau}{2} = 0$$

We know that good estimate of the shape in this case is

$$X_3 = \frac{\tau}{2}\,t^{3/4}$$

so that

$$\theta = \frac{3\tau}{8}\,t^{-\frac{1}{4}}$$

Therefore the influence functions in this case can be calculated as

$$\lambda_1{}^{\phi} = 1\ ,\ \lambda_3{}^{\phi} = \frac{27}{128}\,(1 - t^{\frac{1}{4}})$$

$$\lambda_1{}^{\psi} = 0\ ,\ \lambda_3{}^{\psi} = 1$$

Also

$$I_{\phi\phi} = \left(\frac{27\,\pi^3}{128}\right)^2,\ I_{\psi\psi} = 1\ ,\ I_{\psi\phi} = \frac{27\,\tau^3}{128}$$

Therefore $I_{\phi\phi}\,I_{\psi\psi} - I^2_{\psi\phi} = 0$ and thus verifying that the optimum shape in this case is $X_3 = \dfrac{\tau}{2}\,t^{3/4}$.

Case (*ii*)—(d, S) *prescribed*

Since in this case volume is free, the equation (8) can be ignored and the system equations are

$$\left.\begin{aligned} f_1 &\equiv \dot{X}_1 = X_3\,\theta^3 \\ f_3 &\equiv \dot{X}_3 = \theta \\ f_4 &\equiv \dot{X}_4 = X_3 \end{aligned}\right\} \quad X_1\,(0) = X_3\,(0) = X_4\,(0) = 0 \tag{12}$$

with

$$\phi(1) \equiv \frac{X_1(1)}{X^2_3(1)} \quad \text{and} \quad \psi(1) \equiv \frac{X_4(1)}{X^2_3(1)} - a = 0$$

We take the initial estimate of the shape as $X_3 = \frac{1}{2a} t$ so that $\theta = \frac{1}{2a}$

The influence functions in this case can be calculated as

$$\lambda\phi_1 = \frac{1}{8a^3}, \lambda\phi_3 = -\frac{t}{2a}, \lambda\phi_4 = 0$$

$$\lambda\psi_1 = 0, \lambda\psi_3 = -4a^2 t, \lambda\psi_4 = \frac{1}{8a^3}$$

Here the values of the integrals corresponding to step (v) of the algorithm are

$$I_{\phi\phi} = \frac{1}{3a^2}, I_{\psi\psi} = \frac{16}{3} a^4, I_{\phi\psi} = -\frac{4a}{3}$$

so that

$$I_{\phi\phi} I_{\psi\psi} - I^2_{\phi\psi} = 0$$

and this shows that $X_3 = \frac{1}{2a} t$ is the optimizing curve.

Case (iii)—(d, V) prescribed

In this case since surface area is free, the system equations are

$$\left.\begin{aligned} f_1 &\equiv \dot{X}_1 = X_3\theta^3 \\ f_2 &\equiv \dot{X}_2 = X^2_3 \\ f_3 &\equiv \dot{X}_3 = \theta \end{aligned}\right\} X_1(0) = X_2(0) = X_3(0) = 0 \qquad (13)$$

$$\text{with } \phi(1) \equiv \frac{X_1(1)}{X^2_3(1)} \text{ and } \psi(1) \equiv \frac{X_2(1)}{X_3^3(1)} - \beta = 0$$

If we take the extremizing curve as $X_3 = \frac{1}{4\beta} t^{3/2}$ then in that case $\theta = \frac{3}{8\beta} t^{\frac{1}{2}}$.

Therefore, the influence functions in this case are

$$\lambda\phi_1 = 16\beta^2, \ \lambda\phi_2 = 0, \ \lambda\phi_3 = -\frac{27}{64\beta} + \frac{27}{32\beta}\left(1 - t^{5/2}\right)$$

$$\lambda\psi_1 = 0, \quad \lambda\psi_2 = 64\beta^3, \lambda\psi_3 = -12\beta^2 + \frac{64}{5}.\beta^2\left(1 - t^{5/2}\right)$$

The three integrals corresponding to step (v) are as follows :

$$I_{\phi\phi} = \frac{7047}{28672\beta^2}, I_{\psi\psi} = \frac{464\beta^4}{21}, I_{\phi\psi} = -\frac{261}{112}\beta$$

so that

$$I_{\phi\phi} I_{\psi\psi} - I^2_{\psi\phi} = 0$$

Here again we see that the opitmizing curve can be accurately represented by $X_3 = \frac{1}{4\beta} t^{3/2}$.

Case (iv)—(S, l) are prescribed

Here in this case since the volume is free, equation (8) can be ignored from the analysis and the system equations are

$$\left.\begin{aligned} f_1 &\equiv X_1 = X_3\,\theta^3 \\ f_3 &\equiv \dot{X}_3 = \rho \\ f_4 &\equiv \dot{X}_4 = X_3 \end{aligned}\right\} \quad X_1(0) = X_3(0) = X_4(0) = 0 \tag{14}$$

with $\phi(1) \equiv X_1(1)$ and $\psi(1) \equiv X_4(1) - \gamma = 0$.

Here we estimate that the optimizing curve is $X_3 = \gamma(n+1)t^n$ so that $\theta = \gamma n(n+1)t^{n-1}$

The influence functions are given by

$$\lambda_1^\phi = 1,\ \lambda_3^\phi = 8\gamma^3 (n+1)^3 (1 - t^{3n-2}),\ \lambda_4^\phi = 0$$

$$\lambda_1^\psi = 0,\ \lambda_3^\psi = (1-t) \qquad ,\ \lambda_4^\psi = 1$$

Therefore, the values of $I_{\phi\phi}$, $I_{\psi\psi}$ and $I_{\phi\psi}$ can be obtained as

$$I_{\phi\phi} = \left\{ \frac{2\gamma^3 n^2 (n+1)^3 (4n-3)}{(3n-2)} \right\}^2 \left[\frac{1}{6n-3} + \frac{n^2}{4(4n-3)^2} + \frac{n}{(4n-3)(3n-1)} \right]$$

$$I_{\psi\psi} = \tfrac{1}{3}$$

$$I_{\phi\psi} = \frac{2\gamma^3 (n+1)^3 (4n-3)}{(3n-2)} \left[\frac{n}{4(4n-3)} + \frac{1}{3n(n-1)} \right]$$

so that

$$I_{\phi\phi} I_{\psi\psi} - I^2_{\psi\phi} = \left\{ \frac{2\gamma^3 n^2 (n+1)^3 (4n-3)}{(3n-2)} \right\}^2 \left[\left\{ \frac{1}{9(2n-1)} + \frac{n^2}{12(4n-3)^2} + \right.\right.$$

$$\left.\left. + \frac{n}{3(4n-3)(3n-2)} \right\} - \left\{ \frac{n}{4(4n-3)} + \frac{1}{3n(n-1)} \right\} \right]$$

The value of n in this case is 0·6404 and for this value of n the quantity $I_{\phi\phi} I_{\psi\psi} - I^2_{\phi\psi} = 0$ confirming that the power law body $x_3 = 1{\cdot}6404\,\gamma t^{0\cdot6404}$ is a good estimate of the optimum shape.

Case (v)—(l, V) prescribed

In this case the surface area is free so the system is described by the equations

$$\left.\begin{aligned} \dot{X}_1 &= X_3\theta^3 \\ \dot{X}_3 &= X_3^2 \\ \dot{X}_3 &= \theta \end{aligned}\right\} \quad X_1(0) = X_2(0) = X_3(0) = 0 \tag{15}$$

with $\phi(1) \equiv X_1(1)$ and $\psi(1) \equiv X_2(1) - \mu = 0$.

Here again we approximate the optimal shape by $X_3 = \sqrt{\mu(2n+1)}\,t^n$ so that the influence functions are represented by

$$\lambda_1^\phi = 1,\ \lambda_2^\phi = 0,\ \lambda_3^\phi = \frac{n^3 (2n+1)^{3/2} \mu^{3/2}}{(3n-1)} (1-t)^{3n-2}$$

$$\lambda_1^\psi = 0,\ \lambda_2^\psi = 1,\ \lambda_3^\psi = \frac{2\mu^{1/2} (2n+1)^{1/2}}{(n+1)} (1-t)^{n+1}$$

and thus the values of the integrals $I_{\phi\phi}$, $I_{\psi\phi}$ and $I_{\psi\psi}$ are given by

$$I_{\phi\phi} = \frac{4\mu^3 n^4 (2n+1)^3 (4n-3)^2}{(3n-2)} \left[\frac{1}{3(2n-1)} + \frac{n^2}{4(4n-3)^2} + \frac{n}{(4n-3)(3n-1)} \right]$$

$$I_{\psi\psi} = \frac{8\mu (2n+1)}{(n+2)}$$

$$I_{\phi\psi} = \frac{2\mu^2 n^2 (2n+1)^2 (4n-3)}{(3n-2)} \left[\frac{1}{2n(3n-1)} + \frac{n}{(4n-3)(n+2)} \right]$$

Therefore

$$I_{\phi\phi} I_{\psi\psi} - I^2_{\phi\psi} = \left[\frac{4\mu^2 n^2 (2n+1)^2 (4n-3)}{(3n-2)} \right]^2 \left[\left\{ \frac{2}{(2n-3)(n+2)} \right\} \left\{ \frac{1}{3(2n-1)} + \right. \right.$$

$$\left. \left. + \frac{n^2}{4(4n-3)^2} + \frac{n}{(4n-3)(3n+1)} \right\} - \left\{ \frac{1}{8n(3n-1)} + \frac{n}{4(4n-3)(n+2)} \right\}^2 \right]$$

In this case the value of n can be obtained[3] as $n = 0\cdot6594$. For this value of n, we can easily verify that $I_{\phi\phi} I_{\psi\psi} - I^2_{\phi\psi} = 0$ confirming that $X_3 = \sqrt{2\cdot3188\mu}\; t^{0\cdot6594}$ is a good estimate of the optimum shape.

Case (vi)—(S, V) prescribed

In this case the system of equation are

$$\left.\begin{aligned} f_1 &\equiv \dot{X}_1 = X_3\,\theta^3 \\ f_2 &\equiv \dot{X}_2 = X_3^2 \\ f_3 &\equiv \dot{X}_3 = \theta \\ f_4 &\equiv \dot{X}_4 = X_3 \end{aligned}\right\} \quad X_1(0) = X_2(0) = X_3(0) = X_4(0) = 0 \qquad (16)$$

with $\phi(1) \equiv \dfrac{X_1(1)}{X_4(1)}$ and $\psi(1) \equiv \dfrac{X_2^2(1)}{X_4^3(1)} - \nu = 0.$

Here we take the extremising curve as

$$X_3 = \frac{\tau}{2} \left\{ 1 - (1-t)^{3/2} \right\}$$

so that

$$\theta = \frac{3}{4}\, \tau (1-t)^{\frac{1}{2}}$$

The corresponding influence functions are then

$$\lambda^{\phi} = \frac{1}{X_4(1)},\ \lambda_2^{\phi} = 0,\ \lambda_3^{\phi} = \frac{27\tau^3}{160 X_4(1)} (1-t)^{5/2} - \frac{X_1(1)}{X_4^2(1)} (1-t),\ \lambda_4^{\phi} - \frac{X_1(1)}{X_4^2(1)}$$

$$\lambda^{\psi} = 0,\ \lambda_2^{\psi} = \frac{2X_2(1)}{X_4^3(1)},\ \lambda^{\psi} = \left\{ \frac{2TX_2(1)}{X_4^3(1)} - \frac{3X_2^2(1)}{X_4^4(1)} \right\} (1-t) -$$

$$- \frac{4}{5} T \frac{X_2(1)}{X_4^3(1)} (1-t)^{5/2},\ \lambda_4^{\psi} = -\frac{3X_2^2(1)}{X_4^4(1)}$$

The three quantities $I_{\phi\phi}$, $I_{\psi\psi}$ and $I_{\phi\psi}$ can therefore be calculated as

$$I_{\phi\phi} = \frac{A^2}{3} + \frac{B^2}{6} + \frac{4AB}{9}$$

$$I_{\psi\psi} = \frac{C^2}{3} + \frac{D^2}{6} - \frac{4CD}{9}$$

$$I_{\phi\psi} = \frac{AC}{3} + \frac{2}{9} (BC - AD) - \frac{BD}{6}$$

where

$$A = \frac{315}{128} \tau^2, \quad B = - \frac{9}{4} \tau^2 , \; C = \frac{175}{48} , \; D = \frac{10}{3}$$

so that we find that

$$I_{\phi\phi} \; I_{\psi\psi} - I^2_{\phi\psi} = \frac{1}{162} (AD + BC)^2 = 0$$

which confirms that the optimising curve in this case is

$$X_3 = \frac{T}{2} \left\{ 1 - (1 - t)^{3/2} \right\} \text{ or } X_3 = \frac{16\nu}{15} \left\{ 1 - (1 - t)^{3/2} \right\}$$

CONCLUSION

The above demonstrates the validity of the steepest descent method for solving minimum drag problems. It is established that this method is a useful tool for aerodynamic problems and can be utilised for finding optimum shapes for more complicated situations where analytical solutions are not known .

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