ON THE FORMAL SOLUTION OF DUAL INTEGRAL EQUATIONS OF TWO VARIABLES

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The formal solution of certain dual integral equations involving the H-function of two variables as kernels by fractional integration has been obtained. The given dual integral equations have been transformed, by the application of fractional integration operators, into two others with a common kernel and the problem is then reduced to solving an integral equation.

We give the definition of H-function of two variables due to Verma[1] from which G-function of two variables and other functions as particular cases are deduced. The H-function of two variables

$$H^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{n+p,\,[t+\nu_1,\,t_1+\nu_2],\,s,\,[q+m_1,\,q'+m_2]}\left[\begin{array}{c|l} x & (\epsilon_{n+p}, a_{n+p}) \\ & (\gamma_{t+\nu_1}, r_{t+\nu_1}), (\gamma'_{t_1+\nu_2}, r'_{t_1+\nu_2}) \\ & (\delta_s, d_s) \\ y & (\beta_{q+m_1}, b_{q+m_1})\,;\,(\beta'_{q'+m_2}, b'_{q'+m_2}) \end{array}\right] \tag{1}$$

is defined by the double contour integral

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi\{A(\xi+\eta)\}\,\Psi\{B\xi, c\eta\}\, x^{\xi}\, y^{\eta}\, d\xi\, d\eta \tag{2}$$

where

$$\Phi\{A(\xi+\eta)\} = \frac{\prod_{J=1}^{n}\Gamma(1-\epsilon_J+a_J\xi+a_J\eta)}{\prod_{J=1}^{p}\Gamma(\epsilon_{J+n}-a_{J+n}\xi-a_{J+n}\eta)\prod_{J=1}^{s}\Gamma(\delta_J+d_J\xi+d_J\eta)}$$

$$\Psi\{B\xi, C\eta\} = \frac{\prod_{J=1}^{m_1}\Gamma(\beta_J-b_J\xi)\prod_{J=1}^{\nu_1}\Gamma(\gamma_J+r_J\xi)}{\prod_{J=1}^{q}\Gamma(1-\beta_{J+m_1}+b_{J+m_1}\xi)}\,.$$

$$\cdot\frac{\prod_{J=1}^{m_2}\Gamma(\beta'_J-b'_J\eta)}{\prod_{J=1}^{t}\Gamma(1-\gamma_{\nu_1+J}+\gamma_{\nu_2+J}\xi)\prod_{J=1}^{q_1}\Gamma(1-\beta'_{m_2+J}+b'_{m_2+J}\eta)}\,\cdot$$

$$\cdot\frac{\prod_{J=1}^{\nu_2}\Gamma(\gamma'_J+r'_J\eta)}{\prod_{J=1}^{t_1}\Gamma(1-\gamma'_{\nu_2+J}-r'_{\nu_2+J}\eta)}\,.$$

and

$$0\leqslant n\leqslant p,\ 1\leqslant m_1\leqslant q,\ 1\leqslant m_2\leqslant q',\ 0\leqslant \nu_1\leqslant t,\ 0\leqslant\nu_2\leqslant t_1$$

$$(a_r)\equiv a_1, a_2, \ldots\ldots, a_r\,;\,(a_{r+1,s}) = a_{r+1}, a_{r+2}, \ldots\ldots\ldots, a_s.$$

The sequence of parameters (β_{m_1}, b_{m_1}) (β'_{m_2}, b_{m_2}) $(\gamma_{\nu_1}, r_{\nu_1})$ $(\gamma'_{\nu_2}, r'_{\nu_2})$ and (ϵ_n, a_n) are such that all the poles of the integrand are simple, the paths of integration are intended, if necessary, in such a manner that all the poles of $\Gamma(\beta_J - b_J\xi)$, $(J=1,2,..,m_1)$ and $\Gamma(\beta'_K - b_K\eta)$ $(K=1,2,\ldots m_2)$ lie to the right and those of $\Gamma(\gamma_J + r_J\xi)$, $(J=1,2,\ldots,\nu_1)$, $\Gamma(\gamma'_K + r'_K\eta)$ $(K=1,2,\ldots,\nu_2)$ and $\Gamma(1-\epsilon_J + a_J\xi + a_J\eta)$ $(J=1,2,\ldots. n)$ lie to the left of the imaginary axis.

The integral converges if

$$\begin{aligned} p+q+s+t &< 2(m_1+\nu_1+n) \\ p+q'_1+s+t &< 2(m_2+\nu_2+n) \end{aligned} \tag{3}$$

and

$$|\arg x| < \pi[m_1+\nu_1+n-\tfrac{1}{2}(p+q+s+t)]$$
$$|\arg y| < \pi[m_2+\nu_2+n-\tfrac{1}{2}(p+q'+s+t_1)]$$

Whenever there is no confusion with regard to the parameters, the contracted notation

$$H^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,[t+\nu_1,t_1+\nu_2],s,[q+m_1+q'+m_2]}\begin{bmatrix} x \\ y \end{bmatrix} \text{ or } H\begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

will be used to denote (1).

The dual integral equations to be discussed here are as follows:

$$\int_0^\infty \int_0^\infty H^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,[t+\nu_1,t_1+\nu_2],0,[q+m_1+q'+m_2]}\left[\begin{matrix} xu \\ yu' \end{matrix}\left|\begin{matrix} (\epsilon_{p+n}, a_{p+n}) \\ (\gamma_{t+\nu_1}, r_{t+\nu_1})\ (\gamma'_{t_1'+\nu_2}, r'_{t_1+\nu_2}) \\ \cdots \\ (\beta_{q+m_1}, b_{q+m_1})(\beta'_{q'+m_2}, b'_{q'+m_2}) \end{matrix}\right.\right].$$
$$\cdot\, f(u,u')\,du\,du' = \phi(x,y),\ 0<x<1,\ 0<y<1 \tag{5}$$

$$\int_0^\infty \int_0^\infty H^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,[t+\nu_1,t_1+\nu_2],0,[q+m_1,q'+m_2]}\left[\begin{matrix} xu \\ yu' \end{matrix}\left|\begin{matrix} (\epsilon_{p+n}, a_{p+n}) \\ (c_{t+\nu_1}, r_{t+\nu_1});\ (c'_{t_1+\nu_2}, r'_{t_1+\nu_2}) \\ \cdots \\ (d_{q+m_1}, b_{q+m_1});\ (d'_{q'+m_2}, b'_{q'+m_2}) \end{matrix}\right.\right].$$
$$\cdot\, f(u,u')\,du\,du' = \psi(x,y),\ x>1,\ y>1 \tag{6}$$

where $\phi(x,y)$ and $\psi(x,y)$ are given and $f(x,y)$ is to be determined. We assumed that the H-function of (1) satisfies all the conditions given above with (β_J), (β'_J), replaced by (d_J), (d'_J), respectively and (γ_K), (γ'_K) replaced by (c_K), (c'_K) respectively for $(J=1,2,\ldots,m_1+q)$ or (m_2+q) and $(J=1,2,\ldots,\nu_1+t)$

In this article we use the Mellin type of double integral, *viz.*

***Theorem*:**

If, in the strips $\alpha<\sigma<\beta$ and $a<\gamma<b$.

(*i*) the function of two complex variables $f(r,s)$ is regular,

(*ii*) the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\sigma+it, \gamma+i\tau)|\,dt\,d\tau$$

is absolutely convergent,

(*iii*) $|f(r,s)| \to 0$ as t and τ approach infinitely independently, and if for positive x and y

$$g(x,y) = \frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-r}\, y^{-s}\, f(r,s)\,dr\,ds \tag{7}$$

then

$$f(r,s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} g(x,y)\, dx\, dy \tag{8}$$

which is due to Reed[2]. We may rewrite (7) and (8) in the form

If

$$M\{f(x,y)\} = F(r,s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} g(x,y)\, dx\, dy \tag{9}$$

then

$$M^{-1}\{F(r,s)\} = g(x,y) = \frac{1}{(2\pi i)^2} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-r} y^{-s} F(r,s)\, dr\, ds \tag{10}$$

Then Parseval theorem of the Mellin transform has been restated Fox[3] in the following form which will be formally useful in our investigations.

If

$$M\{h(u,u')\} = H(r,s)$$

and

$$M\{f(xu, yu')\} = x^{-r} y^{-s} F(r,s) \tag{11}$$

where

$$M\{f(u,u')\} = F(r,s)$$

then

$$\int_0^\infty \int_0^\infty h(xu, yu') f(u,u')\, du\, du'$$

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} x^{-r} y^{-s} H(r,s) F(1-r, 1-s)\, dr\, ds \tag{12}$$

From (2) and (10) it is easily seen that

$$M\left\{ H^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,[t+\nu_1,t_1+\nu_2],0,[q+m_1,q'+m_2]} \left[\begin{array}{c|c} x & (\epsilon_{p+n}, a_{p+n}) \\ & (\gamma_{t+\nu_1}, r_{t+\nu_1}) ; (\gamma'_{t_1+\nu_2}, r'_{t_1+\nu_2}) \\ y & \cdots \quad \cdots \quad \cdots \quad \cdots \\ & (\beta_{q+m_1}, b_{q+m_1}) ; (\beta'_{q'+m_2}, b'_{q'+m_2}) \end{array} \right] \right\}$$

$$= \frac{\prod_{J=1}^{m_1} \Gamma(\beta_J + b_J \xi) \prod_{J=1}^{\nu_1} \Gamma(\gamma_J - r_J \xi) \prod_{J=1}^{m_2} \Gamma(\beta'_J + b'_J \eta) \prod_{J=1}^{\nu_2} \Gamma(\gamma'_J - r'_J \eta)}{\prod_{J=1}^{q} \Gamma(1 - \beta_{m_1+J} - b_{m_1+J} \xi) \prod_{J=1}^{t} \Gamma(1 - \gamma_{\nu_1+J} + r_{\nu_1+J} \xi) \prod_{J=1}^{q_1} \Gamma(1 - \beta'_{J+m_2} - \beta'_{J+m_2} \eta)} \cdot$$

$$\cdot \frac{\prod_{J=1}^{n} \Gamma(1 - \epsilon_J - a_J \xi - a_J \eta)}{\prod_{J=1}^{t_1} \Gamma(1 - \gamma'_{\nu_2+J} + r'_{\nu_2+J} \eta) \prod_{J=1}^{p} \Gamma(\epsilon_{n+J} + a_{n+J} \xi + a_{n+J} \eta)} . \tag{13}$$

On using $M\{f(u,u')\} = F(\xi,\eta)$ and applying (12) to (4) and (5) we find that

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{J=1}^{n}\Gamma(1-\epsilon_J-a_J\,\xi-a_J\,\eta)\prod_{J=1}^{m_1}\Gamma(\beta_J+b_J\,\xi)}{\prod_{J=1}^{p}\Gamma(\epsilon_{J+n}+a_{J+n}\xi+a_{J+n}\eta)\prod_{J=1}^{q}\Gamma(1-\beta_{m_1+J}-b_{m_1+J})}\;\cdot$$

$$\cdot\;\frac{\prod_{J=1}^{\nu_1}\Gamma(\gamma_J-r_J\,\xi)\prod_{J=1}^{m_2}\Gamma(\beta'_J+b'_J\,\eta)\prod_{J=1}^{\nu_2}\Gamma(\gamma'_J-r'_J\,\eta)}{\prod_{J=1}^{t}\Gamma(1-\gamma_{\nu_1+J}+r_{\nu_1+J}\,\xi)\prod_{J=1}^{q'}\Gamma(1-\beta'_{m_2+J}-b'_{m_2+J}\,\eta)}\;\cdot$$

$$\cdot\;\frac{x^{-\xi}\,y^{-\eta}\,\Gamma(1-\xi,1-\eta)}{\prod_{J=1}^{t_1}\Gamma(1-\gamma'_{\nu_2+J}+r'_{\nu_2+J}\,\eta)}\,d\xi\,d\eta=\phi(x,y),\ \text{where } 0<x<1,\ 0<y<1, \tag{14}$$

and

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{J=1}^{n}\Gamma(1-\epsilon_J-a_J\,\xi-a_J\,\eta)\prod_{J=1}^{m_1}\Gamma(d_J+b_J\,\xi)}{\prod_{J=1}^{p}\Gamma(\epsilon_{J+n}+a_{J+n}\xi+a_{J+n}\eta)\prod_{J=1}^{q}\Gamma(1-d_{J+m_1}-b_{J+m_1}\xi)}\;\cdot$$

$$\cdot\;\frac{\prod_{J=1}^{\nu_1}\Gamma(c_J-r_J\,\xi)\prod_{J=1}^{m_2}\Gamma(d'_J+b'_J\,\eta)\prod_{J=1}^{\nu_2}\Gamma(c'_J-r'_J\,\eta)\,x^{-\xi}\,y^{-\eta}\,\Gamma(1-\xi,1-\eta)}{\prod_{J=1}^{t}\Gamma(1-c_{\nu_1+J}+r_{\nu_1+J}\,\xi)\prod_{J=1}^{q'}\Gamma(1-d'_{m_2+J}-b'_{m_2+J}\,\eta)\prod_{J=1}^{t_1}\Gamma(1-c'_{\nu_2+J}+r'_{\nu_2+J}\,\eta)}\,d\xi\,d\eta$$

$$=\psi(x,y),\ \text{where } x>1,\ y>1,\ \text{by virtue of (13)}. \tag{15}$$

In this section we transform the equations (14) and (15) into two others with a common kernel by the application of fractional operators. Our main aim is to transform

$$\frac{\prod_{J=1}^{\nu_1}\Gamma(\gamma_J-r_J\,\xi)\prod_{J=1}^{\nu_2}\Gamma(\gamma'_J-r'_J\,\eta)}{\prod_{J=1}^{q}\Gamma(1-\beta_{m_1+J}-b_{m_1+J}\,\xi)\prod_{J=1}^{q'}\Gamma(1-\beta'_{m_2+J}-b'_{m_2+J}\,\eta)}\ \text{of (14)} \tag{16}$$

into

$$\frac{\prod_{J=1}^{\nu_1}\Gamma(c_J-r_J\,\xi)\prod_{J=1}^{\nu_2}\Gamma(c'_J-r'_J\,\eta)}{\prod_{J=1}^{q}\Gamma(1-d_{m_1+J}-b_{m_1+J}\,\xi)\prod_{J=1}^{q'}\Gamma(1-d'_{m_2+J}-b'_{m_2+J}\,\eta)}\ \text{of (15)}$$

and

$$\frac{\prod_{J=1}^{m_1}\Gamma(\beta_J+b_J\,\xi)\prod_{J=1}^{m_2}\Gamma(\beta'_J+b'_J\,\eta)}{\prod_{J=1}^{t}\Gamma(1-\gamma_{\nu_1+J}+r_{\nu_1+J}\,\xi)\prod_{J=1}^{t_1}\Gamma(1-\gamma'_{\nu_2+J}+r'_{\nu_2+J}\,\eta)}\ \text{of (14)}$$

into

$$\frac{\prod_{J=1}^{m_2}\Gamma(d'_J+b'_J\,\eta)\prod_{J=1}^{m_1}\Gamma(d_J+b_J\,\xi)}{\prod_{J=1}^{t_1}\Gamma(1-c'_{\nu_2+J}+r'_{\nu_2+J}\,\eta)\prod_{J=1}^{t}\Gamma(1-c_{\nu_1+J}+r_{\nu_1+J}\,\xi)}\ \text{of (15)} \tag{17}$$

In making these transformations we will make use of the fractional integration operators for two variables, similar to the fractional integration operators for one variable defined by Erdelyi[4]

$$\tau_{11}\left[\alpha,\beta,\alpha',\beta':r,s:\omega(x,y)\right]=\frac{rs}{\Gamma(\alpha)\,\Gamma(\alpha')}\,x^{-r\alpha+r-\beta-1}\,.$$

$$.\,y^{-s\alpha'+s-\beta'-1}\int_0^{\infty}\int_0^{\infty}\left(x^r-v^r\right)^{\alpha-1}\left(y^s-v'^s\right)^{\alpha-1}v^{\alpha}\,v'^{\beta'}\,\omega(v,v')\,dv\,.\,dv' \tag{18}$$

$$R_{11}\left[\alpha,\beta,\alpha';\beta':r,s:\omega(x,y)\right]=\frac{rs}{\Gamma(\alpha)\,\Gamma(\alpha')}\,x^{\beta}\,y^{\beta'}\,.$$

$$.\int_x^{\infty}\int_y^{\infty}(v^r-x^r)^{\alpha-1}(v'^s-y^s)^{\alpha'-1}\,v^{-\beta-\alpha r+r-1}\,v'^{-\beta'-s\alpha'+s-1}\,\omega(v,v')\,dv\,dv' \tag{19}$$

In the contracted forms we write

$$\tau\left[(\gamma_J-c_J),(\gamma'_J-c'_J);(c_J r_J^{-1}-1),(c'_J r'^{-1}_J-1);r_J^{-1},r'^{-1}_J:\omega(x,y)\right]=\tau_J\left[\omega(x,y)\right] \tag{20}$$

$$\tau\Big[(\beta_{m_1+k}-d_{m_1+k}),(\beta'_{m_2+k}-d'_{m_2+k});(b^{-1}_{m_1+k}-\beta^{-1}_{m_1+k}\,b'_{m_1+k}-1),$$
$$(b'^{-1}_{m_2+k}-\beta'^{-1}_{m_2+k}\,b'_{m_2+k}-1);b^{-1}_{m_1+k},b'^{-1}_{m_2+k};\omega(x,y)\Big]=\tau_k\left[\omega(x,y)\right] \tag{21}$$

$$R\left[(B_l-d_l),(\beta'_l-d'_l);d_l\,b_l^{-1},d'_l\,b'^{-1}_l;\,b_l^{-1}\,b'^{-1}_l;\;\omega(x,y)\right]=R_l\left[\omega(x,y)\right] \tag{22}$$

$$R\Big[(\gamma_{\nu_1+h}-c_{\nu_1+h}),(\gamma'_{\nu_2+h}-c'_{\nu_2+h}):(r'^{-1}_{\nu_1+h}-\gamma_{\nu_1+h}\,r^{-1}_{\nu_1+h}),$$
$$(r'^{-1}_{\nu_2+h}-\gamma'_{\nu_2+h}\,r'^{-1}_{\nu_2+h}):r^{-1}_{\nu_1+h},r'^{-1}_{\nu_2+h}:\omega(x,y)\Big]=R^*_h\left[\omega(x,y)\right] \tag{23}$$

We now proceed to make the first transformation in (14). We replace x and y by v and v' respectively, multiply by

$$v'^{\,c'_{\nu_2}e'_{\nu_2}-1}\left(x^{e_{\nu_2}}-v'^{\,e_{\nu_2}}\right)^{\gamma_{\nu_2}-c'_{\nu_2}-1}$$

where $e_{\nu_1}=1/r_{\nu_1}$ and $e_{\nu_2}=1/r_{\nu_2}$ and integrate under the integral sign with respect to v and v' from 0 to x and 0 to y respectively where $0<x<1,\ 0<y<1$ we obtain

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{J=1}^{n}\Gamma(1-{}_J-a_J\,\xi-a_J\,\eta)\prod_{J=1}^{m_1}\Gamma(\beta_J+h_J\,\xi)}{\prod_{J=1}^{p}\Gamma(\epsilon_{J+n}+e_{J+n}\,\xi+a_{J+n}\,\eta)\prod_{J=1}^{q}\Gamma(1-\beta_{J+m_1}-b_{J+m_1}\xi)}\,.$$

$$.\,\frac{\prod_{J=1}^{m_2}\Gamma(\beta'_J-b'_J\,\eta)\prod_{J=1}^{\nu_1-1}\Gamma(\gamma_J-r_J\,\xi)\prod_{J=1}^{\nu_2-1}\Gamma(\gamma'_J-r'_J\,\eta)}{\prod_{J=1}^{t}\Gamma(1-\gamma_{\nu_1+J}+r_{\nu_1+J}\,\xi)\prod_{J=1}^{q'}\Gamma(1-\beta'_{m_2+J}-b'_{m_2+J}\,\eta)}\,.$$

$$.\,\frac{\Gamma(c_{\nu_1}-r_{\nu_1}\,\xi)\,\Gamma(c'_{\nu_2}-r'_{\nu_2}\,\eta)\,x^{-\xi}\,y^{-\eta}\,\Gamma(1-\xi,1-\eta)}{\prod_{J=1}^{t_1}\Gamma(1-\gamma'_{\nu_2+J}+\gamma'_{\nu_2+J}\,\eta)}\,d\xi\,d\eta$$

$$=\frac{r^{-1}_{\nu_1}\,r^{-1}_{\nu_2}\,x^{r^{-1}_{\nu_1}-r'_{\nu_1}\gamma_{\nu_1}}\,y^{r'^{-1}_{\nu_2}-r'_{\nu_2}\gamma_{\nu_2}}}{\Gamma(\gamma_{\nu_1}-c_{\nu_1})\;\Gamma(\gamma'_{\nu_1}-c'_{\nu_1})}\,.$$

$$\cdot \int_0^x\int_0^y c_{\nu_1} r_{\nu_1}^{-1} v^{-1} \, c'_{\nu_2} r'^{-1}_{\nu_2} (v')^{-1} \left(x^{r_{\nu_1}^{-1}} - v^{r_{\nu_1}^{-1}} \right)^{\gamma_{\nu_1} - c_{\nu_1} - 1} \cdot \left(x^{r_{\nu_2}^{-1}} - v'^{r_{\nu_2}^{-1}} \right)^{\gamma'_{\nu_2} - c'_{\nu_2} - 1} dv\, dv'$$

$$= \tau_{\nu_1\nu_2} \left[\phi\ (v, v') \right] \tag{24}$$

where $0 < x < 1$, $0 < y < 1$ by virtue of (18) and (20).

On transforming (24) successively, for $(k = \nu_1 - 1, \ldots, 2, 1; \nu_2 - 1, \ldots, 2, 1)$ by the application of operators τ_{kk} and τ^*_{kk} $(k = q, \ldots, 3, 2, 1; k = q', q' - 1, \ldots 3, 2, 1)$

We finally get

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{J=1}^{n} \Gamma(1 - \epsilon_J - a_J\,\xi - a_J\,\eta) \prod_{J=1}^{m_1} \Gamma(\beta_J + b_J\,\xi)}{\prod_{J=1}^{p} \Gamma(\epsilon_{n+J} + a_{n+J}\,\xi + a_{n+J}\,\eta) \prod_{J=1}^{t} \Gamma(1 - \gamma_{\nu_1+J} + r_{\nu_1+J}\,\xi)}$$

$$\cdot \frac{\prod_{J=1}^{\nu_1} \Gamma(c_J - r_J\,\xi) \prod_{J=1}^{\nu_2} \Gamma(c'_J - r'_J\,\eta) \prod_{J=1}^{m_2} \Gamma(\beta'_J + b'_J\,\eta)}{\prod_{J=1}^{q} \Gamma(1 - d_{J+m_1} - b_{J+m_1}\,\xi) \prod_{J=1}^{q'} \Gamma(1 - d'_{m_2+J} - b'_{m_2+J}\,\eta)}$$

$$\cdot \frac{x^{-\xi}\, y^{-\eta}\, F(1 - \xi, 1 - \eta)}{\prod_{J=1}^{t_1} \Gamma(1 - \gamma'_{\nu_2+J} + r'_{\nu_2+J}\,\eta)}\, d\xi\, d\eta$$

$$= \tau^*_{11} \left[\tau^*_{22}, \ldots, \tau^*_{qq'}, \tau_{11}, \ldots, \tau_{\nu_1\nu_2} \left\{ \phi\ (x, y) \right\} \ldots \right], 0 < x < 1, 0 < y < 1. \tag{25}$$

In a similar manner by the application of the operators R_l and R_h^* given in (22) and (23) respectively for $(l = m_1, m_1 - 1, \ldots, 3, 2, 1;\ m_2, m_2 - 1, \ldots, 3, 2, 1$ and $(h = t, t - 1, \ldots 3, 2, 1;\ t_1, t_1 - 1, \ldots, 3, 2, 1)$ to (15) it can be easily seen that it transforms into the desired form

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{J=1}^{n} \Gamma(1 - \epsilon_J - a_J\,\xi - a_J\,\eta) \prod_{J=1}^{m_1} \Gamma(\beta_J + b_J\,\xi)}{\prod_{J=1}^{p} \Gamma(\epsilon_{J+n} + a_{J+n}\,\xi + a_{J+n}\,\eta) \prod_{J=1}^{t_1} \Gamma(1 - \gamma_{J+\nu_1} + r_{J+\nu_1}\,\xi)}$$

$$\cdot \frac{\prod_{J=1}^{\nu_1} \Gamma(c_J - r_J\,\xi) \prod_{J=1}^{\nu_2} \Gamma(c'_J - r'_J\,\xi) \prod_{J=1}^{m_2} \Gamma(\beta'_J + b'_J\,\xi)}{\prod_{J=1}^{q} \Gamma(1 - d_{J+m_1} - b_{J+m_1}\,\xi) \prod_{J=1}^{q'} \Gamma(1 - d'_{m_2+J} - b'_{m_2+J}\,\eta)}$$

$$\cdot \frac{x^{-\xi}\, y^{-\eta}\, F(1 - \xi, 1 - \eta)}{\prod_{J=1}^{t_1} \Gamma(1 - \gamma'_{\nu_2+J} + r'_{\nu_2+J}\,\eta)}\, d\xi\, d\eta$$

$$= R^*_{11} \left[R^*_{22} \ldots, R^*_{tt'}, R_{11}, R_{22}, \ldots, R_{m_1 m_2} \left\{ \psi\ (x, y) \right\} \ldots \right], \ x > 1, y > 1. \tag{26}$$

On setting

$$k\ (x, y) = \tau^*_{11} \left[\tau^*_{22}, \ldots, \tau^*_{qq'}\ , \tau_{11}, \ldots \tau_{\nu_1 \nu_2} \left\{ \phi\,(x, y) \right\} \ldots \right], 0 < x < 1, 0 < y < 1.$$

$$= R_{11}^* \left[R^*_{22}, \ldots, R^*_{tt'}, R_{11} \ldots R_{m_1 m_2} \left\{ \psi\,(x, y) \right\} \ldots \right], x > 1, y > 1. \tag{27}$$

(25) and (26) can be put into a compact form

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{J=1}^{n} \Gamma\,(1 - \epsilon_J - a_J\,\xi - a_J\,\eta) \prod_{J=1}^{m_1} \Gamma\,(\beta_J + b_J\,\xi) \prod_{J=1}^{m_2} \Gamma\,(\beta'_J + b'_J\,\eta)}{\prod_{J=1}^{p} \Gamma\,(\epsilon_{J+n} + a_{J+n}\,\xi + a_{J+n}\,\eta) \prod_{J=1}^{t} \Gamma\,(1 - \gamma_{\nu_1+J} + r_{\nu_1+J}\,\xi)} \cdot$$

$$\cdot \frac{\prod_{J=1}^{\nu_1} \Gamma\,(c_J - r_J\,\xi) \prod_{J=1}^{\nu_2} \Gamma\,(c'_J - r'_J\,\eta)}{\prod_{J=1}^{q} \Gamma\,(1 - d_{J+m_1} - b_{J+m_1}\,\xi) \prod_{J=1}^{q'} \Gamma\,(1 - d'_{m_2+J} - b'_{m_2+J}\,\eta)} \cdot$$

$$\cdot \frac{x^{-\xi}\,y^{-\eta}\,F\,(1 - \xi, 1 - \eta)}{\prod_{J=1}^{t_1} \Gamma\,(1 - \gamma'_{\nu_2+J} + r'_{\nu_2+J}\,\eta)}\, d\,\xi\, d\,\eta = k\,(x, y) \tag{28}$$

Equation (28) is the reduction of (14) and (15) to the equation with a common kernel. On treating the kernel of (28) as an unsymmetrical Fourier kernel and following a procedure similar to that of one variable by Fox[3], (28) can be written as

$$f\,(x, y) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\prod_{J=1}^{p} \Gamma\,(\epsilon_{J+n} + 2a_{J+n} - a_{J+n}\,\xi - a_{J+n}\,\eta)}{\prod_{J=1}^{n} \Gamma\,(1 - \epsilon_J - 2a_J + a_J\,\xi + a_J\,\eta)} \cdot$$

$$\frac{\prod_{J=1}^{t} \Gamma\,(1 + r_{\nu_1+J} - \gamma_{\nu_1+J} - r_{\nu_1+J}\,\xi) \prod_{J=1}^{q} \Gamma\,(1 - b_{J+m_1} - d_{J+m_1} + b_{J+m_1}\,\xi)}{\prod_{J=1}^{m_1} \Gamma\,(\beta_J + b_J - b_J\,\xi) \prod_{J=1}^{m_2} \Gamma\,(\beta'_J + b'_J - b'\,\eta)} \cdot$$

$$\cdot \frac{\prod_{J=1}^{q_1} \Gamma\,(1 - b'_{m_2+J} - d'_{m_2+J} + b'_{m_2+J}\,\eta) \prod_{J=1}^{t_1} \Gamma\,(1 + r'_{\nu_2+J} - \gamma'_{\nu_2+J} + r'_{\nu_2+J}\,\eta)}{\prod_{J=1}^{\nu_1} \Gamma\,(c_J - r_J + r_J\,\xi) \prod_{J=1}^{\nu_2} \Gamma\,(c'_J - r'_J + r'_J\,\eta)}$$

$$\cdot\ K\,(1 - \xi, 1 - \eta)\, d\,\xi\, d\,n \tag{29}$$

where

$$M \left\{ k\ (x, y) \right\} = K\,(\xi, \eta)$$

This is the formal solution of (14) and (15).

Applying (12) to (29) we see that

$$f\ (x, y) = \int_0^\infty \int_0^\infty H^{\,p,\,t,\,t_1,\,q,\,q'}_{\,p+n,\,[t+\nu_1,\ t_1+\nu_2],\,0,\,[q+m_1,\,q'+m_2]}$$

$$\left[\begin{array}{c|l} x\ u & (-1-\epsilon_{J+n,p}-2a_{J+n,p},\ a_{J+n,p})(1-\epsilon_{J,n}-2a_{J,n}, a_{J,n}) \\ & (1-b_{J+m_1,q}-d_{J+m_1,q}\ ,\ b_{J+m_1,q})\ (1-b'_{m_2+J,q_1}-d'_{m_2+J,q_1}, b'_{J+m_2,q_1}) \\ y\ v & \cdots\cdots\cdots \\ & (1-r_{\nu_2+J,t}-\gamma_{\nu_2+J,t}, r_{\nu_2+J,t})\ (1-r'_{J+\nu_2,t_1}-\gamma'_{\nu_2+J,t_1}\ ,\ r'_{J+\nu_2,t_1}) \end{array} \right].$$

$$.\ k\,(u, v)\,du\,dv. \qquad (30)$$

It is clear from the nature of this solution that it can be written down by the inspection from (4) and (5).

Written out in full (30) becomes

$$f\ (x, y) = \int_0^1 \int_0^1 H^{\,p,\,t,\,t_1,\,q,\,q'}_{\,p+n,\,[t+\nu_1,\,t_1+\nu_2],\,0,\,[q+m_1,\,q'+m_2]}$$

$$\left[\begin{array}{c|l} x\ u & (-1-\epsilon_{J+n,p}-2a_{J+n,p},\ a_{J+n,p})\ ;\ (1-\epsilon_{J,n}-2a_{J,n}\ ,\ a_{J,n}) \\ & (1-b_{J+m_1,q}-d_{q+m_1,q}\ ,\ b_{J+m_1,q});\ (1-b'_{m_2+J,q'}-d'_{m_2+J,q'}, b'_{m_2+J,q'}) \\ y\ v & \cdots\cdots\cdots \\ & (1-r_{J+\nu_1,t}-\gamma_{\nu_1+J,t}, r_{\nu_1+J,t})\ ;\ (1-r'_{J+\nu_2,t_1}-\gamma'_{\nu_2+J,t_1}\ ,\ r'_{J+\nu_2,\ t_1}) \end{array} \right].$$

$$.\ \tau_{11}^*\left[\ \tau^*_{22},\ldots,\ \tau^*_{qq'}\ ,\ \tau_{11},\ \ldots,\ \tau_{\nu_1\nu_2}\left\{\phi\,(u, u')\right\}\ldots\ \right] du\,du' \ +$$

$$+ \int_1^\infty \int_1^\infty H^{\,p,\,t,\,t_1,\,q,\,q'}_{\,p+n,\,[t+\nu_1,\,t_1+\nu_2],\,0,\,[q+m_1,\,q'+m_2]}$$

$$\left[\begin{array}{c|l} x\ u & (-1-\epsilon_{J+n,p}-2a_{J+n,p}, a_{J+n,p})\ (1-\epsilon_{J,n}-2a_{J,n}\ ,\ a_{J,n}) \\ & (1-b_{J+m_1,q}-d_{J+m_1,q}\ ,\ b_{J+m_1,q})\ (1-b'_{m_2+J,q_1}-d'_{m_2+J,q'}, b'_{m_2+J,q'}) \\ y\ v & \cdots\cdots\cdots \\ & (1-r_{\nu_1+J,t}-\gamma_{\nu_1+J,t}\ ,\ r_{\nu_1+J,t})\ (1-r'_{J+\nu_1,t_1}-\gamma'_{\nu_2+J,t_1}\ ,\ r'_{J+\nu_2,t_1}) \end{array} \right]$$

$$R^*_{11}\left[\ R^*_{22},\ \ldots,\ R^*_{tt_1}\ ,\ R_{11},\ \ldots,\ R_{m_1m_2}\ \left\{\ \psi\,(u\,u'\right\}\ldots\ \right] du\,du' \qquad (31)$$

where the H-function is defined in (1) and (2), τ^*_{ii}, ... τ_{JJ}'s are defined by (21), R^*_{ii} .... R_{JJ}'s are defined by (22) and (23).

Finally we say that these results are interesting in the sense. If we put $a_i = r_k = r'_l = b_m = b'_n = 1$ where $(i = 1, 2, \ldots, p\ ;\ k = 1, 2, \ldots,\ t;\ l = 1, 2, .., t';\ m = 1, 2, ..q;\ n = 1, q')$ we obtain the results of Massed & Srivastava[5].

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