SOURCE FLOW BETWEEN TWO POROUS DISKS

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The flow of a viscous, incompressible fluid between two parallel porous disks when the lower disk is rotating and the upper disk is at rest has been studied. The solution obtained is valid for small values of wall Reynolds number.

Recently Elkouh[1] studied the laminar source flow between parallel porous disks for small wall Reynolds number. The solution obtained therein is in the form of a perturbation from the creeping flow solution. In this paper an effort has been made to extend the problem studied by Elkouh[1] to the case when the lower disk is rotating and the upper disk is at rest.

Similar source flow problems have also been studied previously[2-5].

FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

Let us take the axis of rotation of the disk as $\bar{z}$-axis and let the two disks be situated at $\bar{z} = \pm a$. Consider the flow of an incompressible fluid between a rotating and a fixed porous disk with a source at the centre.

The governing non-dimensional hydro-dynamic equations of motion in cylindrical polar coordinates are

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}, \tag{2}$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \tag{3}$$

and the equation of continuity is

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{4}$$

where the non-dimensional quantities are defined as

$$r = \frac{r}{a}, \; z = \frac{\bar{z}}{a}, \; u = \frac{\bar{u}\, a}{\nu}, \; v = \frac{\bar{v}\, a}{\nu}, \; w = \frac{\bar{w}\, a}{\nu} \text{ and } p = \frac{\bar{p}\, a^2}{\rho\, \nu^2}.$$

Let the lower disk be rotating with angular velocity ω and the strength of the source be Q. The boundary conditions are

$$\left.\begin{aligned} u &= 0 && \text{at } z = \pm 1 \\ v &= Rw\, \alpha\, r && \text{at } z = -1 \\ v &= 0 && \text{at } z = 1 \\ w &= \mp Rw && \text{at } z = \pm 1, \\ &\text{and} \\ \int_{-1}^{+1} u\,dz &= \frac{2\,Re}{r} + Rwr, \end{aligned}\right\} \tag{5}$$

where

α = Rotational Taylor number = $\frac{\omega a}{\nu}$

Re = Reynolds number = $\frac{Q}{4\pi\nu a}$; Rw = Wall Reynolds number = $\frac{Va}{\nu}$ and V is the magnitude of the constant injection velocity at the two disks.

SOLUTION OF THE PROBLEM

Let us define a stream function ψ as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z} \; , w = -\frac{1}{r}\frac{\partial \psi}{\partial r} \; . \tag{6}$$

For small values of Re/r^2, following Elkouh[1], let

$$\psi = \frac{1}{2} r^2 Rw f_{-1}(z) + Re \left[f_0(z) + \left(\frac{Re}{r^2}\right) f_1(z) + \left(\frac{Re}{r^2}\right)^2 f_2(z) + \dots \right] , \tag{7}$$

$$u = \frac{1}{2} r Rw f'_{-1}(z) + \left(\frac{Re}{r}\right) \left[f'_0(z) + \left(\frac{Re}{r^2}\right) f'_1(z) + \left(\frac{Re}{r^2}\right)^2 f'_2(z) + \dots \right] , \tag{8}$$

$$v = r Rw g_{-1}(z) + \frac{Re}{r} \left[g_1(z) + \left(\frac{Re}{r^2}\right) g_3(z) + \left(\frac{Re}{r^2}\right)^2 g_5(z) + \dots \right] , \tag{9}$$

$$w = - Rw f_{-1}(z) + \left[2 \left(\frac{Re}{r^2}\right)^2 f_1(z) + 4 \left(\frac{Re}{r^2}\right)^3 f_2(z) + \dots \right] ; \tag{10}$$

$$p = \frac{1}{4} r^2 Rw h_{-1}(z) + h(z) + Re \left[h_0(z) \log r + \left(\frac{Re}{r^2}\right) h_1(z) + \left(\frac{Re}{r^2}\right)^2 h_2(z) + \dots \right] , \tag{11}$$

where prime denote differentiation with respect to z.

Substituting the values of u, v, w and p from (8) to (11) in (1) to (3) and equating terms of like powers of r on both sides, we get an infinite system of simultaneous ordinary differential equations. The first four systems are

System I

$$f'''_{-1} + Rw \left(f_{-1} f''_{-1} - \frac{1}{2} f'^2_{-1} + 2 g^2_{-1} \right) = h_{-1} ,$$

$$g''_{-1} + Rw (f_{-1} g'_{-1} - f'_{-1} g_{-1}) = 0 ,$$

$$h'_{-1} = 0$$

System II

$$f'''_0 + Rw (f_{-1} f''_0 - 2 g_{-1} g_1) = h_0 ,$$

$$g''_{-1} - Rw\,(2 f'_0\, g_{-1} - f_{-1}\, g'_1) = 0\ ,$$

$$h'_0 = 0$$

System III

$$f'''_1 + Rw\,(f'_{-1} f'_1 + f_{-1} f''_1 - f_1 f''_{-1} + 2\, g_{-1}\, g_3) = -\, g^2_{-1} - 2\, h_1 - f'^2_0\,,$$

$$g''_3 + 2\, Rw\,(f'_{-1}\, g_3 + f_{-1}\, g'_3 - f'_1\, g_{-1} - f_1\, g'_{-1}) = 0\ ,$$

$$h'_1 = 0\ ,$$

System IV

$$f'''_2 + Rw\,(2 f'_{-1} f'_2 - 2 f_2 f''_{-1} + f_{-1} f''_2) = 2 f_1 f''_0 - 4 f'_1 f'_0 - 2 g_1\, g_2 - 2 g_1\, g_5 -$$

$$- \frac{8}{Re} f'_1 - 4 h_2\ ,$$

$$g''_5 - Rw\,(-\,2 f'_{-1}\, g_5 - 3 f'_0\, g_3 + 2 f'_2\, g_{-1} - f_{-1}\, g'_5 + 2 f_1\, g'_1 + 4 f_2\, g'_{-1} + f'_0\, g_3) +$$

$$+ \frac{8}{Re}\, g_3 = 0\ ,$$

$$h'_2 = \frac{2}{Re} f''_1 + 2 \left(\frac{Rw}{Re}\right) (f_{-1} f'_1 + 3 f'_{-1}\, f_1).$$

For small values of Rw, let us expand the function f_n, g_n and h_n in ascending powers of Rw as follows :

$$f_n = \sum_{i=0}^{\infty} Rw^i f_{n,i}\ ;\ g_n = \sum_{i=0}^{\infty} Rw^i\, g_{n,i}\ ;\ \text{and}\ h_n = \sum_{i=0}^{\infty} Rw^i\, h_{n,i} \qquad (12)$$

Substituting the expressions (12) into the system of differential equations I to IV, we get on equating the coefficients of like powers of Rw, another infinite sets of equations. On solving those differential equations under the boundary conditions (5), we get .

$$f'_{-1} = \frac{3}{2} - \frac{3}{2} z^2 + Rw \left[\frac{19}{560} - \frac{117 z^2}{560} + \frac{105}{560} z^4 - \frac{7 z^6}{560} + \alpha^2 \left(- \frac{5 z^4}{120} + \frac{4 z^3}{24} + \right.\right.$$

$$\left.\left. + \frac{3 z^2}{60} - \frac{z}{6} - \frac{1}{120} \right) \right] + Rw^2 \left[- \frac{3288}{2587200} - \frac{6645}{2587200} z^2 + \frac{78540}{2587200} z^4 - \right.$$

$$- \frac{81774}{2587200} z^6 + \frac{13860}{2587200} z^8 - \frac{693}{2587200} z^{10} + \alpha^2 \left(\frac{9}{120960} z^8 - \frac{8}{6720} z^7 + \right.$$

$$+ \frac{63}{5600} z^6 + \frac{6}{720} z^5 - \frac{355}{4800} z^4 - \frac{44}{320} z^3 + \frac{5088105561}{4563820800} z^2 + \frac{73}{560} z -$$

$$\left.\left. - \frac{1767518445}{4563820800} \right) \right]; \qquad (13)$$

$$f'_0 = \frac{3}{2} - \frac{3}{2} z^2 + Rw \left[\frac{17}{280} - \frac{108}{280} z^2 + \frac{105}{280} z^4 - \frac{14}{280} z^6\right] + Rw^2 \left[\frac{5563}{5174400} - \right.$$

$$- \frac{170112}{5174400} z^2 + \frac{542850}{5174400} z^4 - \frac{490644}{5174400} z^6 + \frac{121275}{5174400} z^8 - \frac{8932}{5174400} z^{10} +$$

$$+ \alpha^2 \left(\frac{99 z^8}{60480} + \frac{2056}{403200} z^7 + \frac{483}{50400} z^6 - \frac{6}{96} z^5 + \frac{55}{1200} z^4 + \frac{44}{480} z^3 - \right.$$

$$\left.\left. - \frac{13285}{1209600} z^2 - \frac{3454}{100800} z + \frac{1043}{403200} \right)\right], \quad (14)$$

$$f'_1 = \frac{15}{280} - \frac{99}{280} z^2 + \frac{105}{280} z^4 - \frac{21}{280} z^6 + Rw \left(\frac{2003}{184800} - \frac{20703}{184800} z^2 + \right.$$

$$\left. + \frac{43890}{184800} z^4 - \frac{34650}{184800} z^6 + \frac{10395}{184800} z^8 - \frac{935}{184800} z^{10} \right) + O\,(Rw^2)\;, \quad (15)$$

$$f'_2 = \frac{1}{431200} \left(5301 - 44340\, z^2 + 76230\, z^4 - 52668\, z^6 + 17325\, z^8 - 1848\, z^{10} \right) +$$

$$+ \frac{1}{2100\, Re} \left(93 - 708\, z^2 + 990\, z^4 - 420\, z^6 + 45\, z^8 \right)\;, \quad (16)$$

$$g_{-1} = \frac{\alpha}{2} (1 - z) + Rw \left[\frac{\alpha z^5}{40} - \frac{\alpha z^4}{16} + \frac{3\alpha}{8} z^2 - \frac{\alpha z}{40} - \frac{5\alpha}{16} \right] + Rw^2 \left[\frac{34\alpha}{80640} z^9 - \right.$$

$$- \frac{6\alpha}{8960} z^8 + z^7 \left\{ \frac{4\alpha^3}{10080} - \frac{252\,\alpha}{47040} \right\} + z^6 \left\{ - \frac{\alpha^3}{1260} + \frac{210\alpha}{33600} \right\} +$$

$$+ z^5 \left\{ \frac{106\alpha}{22400} + \frac{11\alpha^3}{2400} \right\} + z^4 \left\{ \frac{8\alpha^3}{1440} - \frac{222\alpha}{13440} \right\} + z^3 \left\{ - \frac{\alpha^3}{80} \right\} +$$

$$+ z^2 \left\{ - \frac{11\alpha^3}{480} - \frac{506\alpha}{2240} \right\} + z \left\{ \frac{41\,\alpha}{201600} + \frac{379\alpha^3}{50400} \right\} + \frac{1061\,\alpha}{4480} +$$

$$\left. + \frac{29\alpha^3}{1440} \right]\;, \quad (17)$$

$$g_1 = Rw \left[- \frac{\alpha z^3}{4} + \frac{3\alpha}{4} z^2 + \frac{3\alpha}{40} z^5 - \frac{\alpha}{8} z^4 + \frac{7\alpha}{40} z - \frac{5\alpha}{8} \right] + Rw^2 \left[\frac{91\alpha}{403200} z^9 - \right.$$

$$- \frac{63\alpha}{31360} z^8 - \frac{693\alpha}{23520} z^7 + \frac{315\alpha}{16800} z^6 + \frac{937\alpha}{11200} z^5 - \frac{321\alpha}{6720} z^4 - \frac{223\alpha}{3160} z^3 -$$

$$\left. - \frac{559\alpha}{1120} z^2 + \frac{20227543\alpha}{111484800} z + \frac{249375}{470400} \alpha \right], \quad (18)$$

$$g_3 = Rw \left[- \frac{19\alpha}{2240} + \frac{16\alpha}{4200} z + \frac{15\alpha}{560} z^2 - \frac{5\alpha}{280} z^3 - \frac{33\alpha}{1120} z^4 + \frac{33\alpha}{1400} z^5 + \frac{7\alpha}{560} z^6 - \right.$$

$$\left. - \frac{3\alpha}{280} z^7 - \frac{3\alpha}{2240} z^8 + \frac{\alpha}{840} z^9 \right] + O\,(Rw^2)\,, \quad (19)$$

$$g_5 = O + O\,(Rw)\;, \quad (20)$$

For $\alpha = 0$, the above results reduce to those obtained by Elkouh[1].

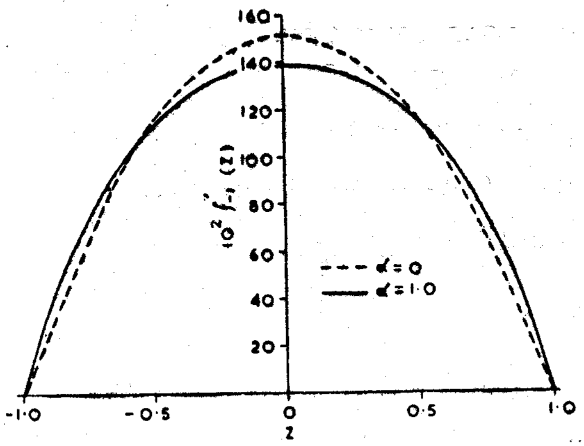

Fig. 1—Variation of $10^2 f'_{-1}(z)$ with z for $Rw = 0{\cdot}5$.

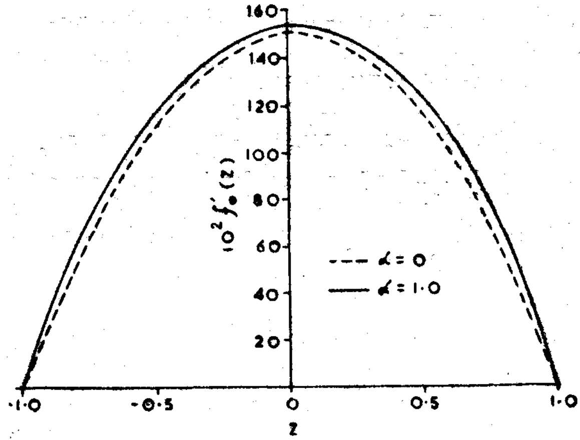

Fig. 2—Variation of $10^2 f'_0(z)$ with z for $Rw = 0{\cdot}5$.

DISCUSSION

Curves have been drawn (Fig. 1 and 2) showing the variation of $f'_{-1}(z)$ and $f'_0(z)$ with z for $\alpha = 0$ and $\alpha = 1$ when $Rw = 0{\cdot}5$ and $Re = 10$. We see from Fig. 1 that the effect of rotation is to increase its $f'_{-1}(z)$ value near the two disks and to damp it in the region $z \simeq (-0{\cdot}5, 0{\cdot}6)$. Again from Fig. 2 it can be easily seen that the effect of rotation on $f'_0(z)$ is to increase it for all values of z. Since Re/r^2 is small, we may conclude that the effect of rotation on the radial velocity is to increase it near the two disks and to impede it in the region midway between the two disks.

Similarly, from Table 1 we infer that the effect of rotation is to increase the transverse velocity.

TABLE 1

$Rw = 0{\cdot}5$

z	$g_{-1}(z)$		$g_1(z)$		$g_3(z)$	
	$\alpha = 0$	$\alpha = 1$	$\alpha = 0$	$\alpha = 1$	$\alpha = 0$	$\alpha = 1$
1·0	0	0	0	0	0	0
0·8	0	0·0697	0	—0·0317	0	—0·0003
0·6	0	0·1434	0	—0·0728	0	—0·0010
0·4	0	0·2229	0	—0·1139	0	—0·0021
0·2	0	0·3128	0	—0·1531	0	—0·0034
0	0	0·4080	0	—0·1687	0	—0·0042
—0·2	0	0·5151	0	—0·1872	0	—0·0040
—0·4	0	0·6314	0	—0·1697	0	—0·0027
—0·6	0	0·7540	0	—0·1262	0	—0·0010
—0·8	0	0·8806	0	—0·0667	0	—0·0006
—1·0	0	1·0000	0	0	0	0

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