SOME INTEGRALS OCCURING IN FRACTURE MECHANICS

K. N. SRIVASTAVA, R.M. PALAIYA & ARATI CHOUDHARY

M. A. College of Technology, Bhopal

(*Received* 19 *January* 1978)

Some integrals occuring in fracture mechanics have been derived in closed form. The results have been used for obtaining stress intensity factor at the edge of interface crack.

Recently, in several problems of fracture mechanics for example, [1, 2, 3, 4 & 5] integrals of the following type have arisen.

$$I_k = \int_0^1 \frac{\cos\left\{\omega \log \dfrac{1+x}{1-x}\right\}}{(\rho^2 - x^2)^{k-\frac{1}{2}}}\, dx \quad , \qquad \rho > 1 \,, \tag{1}$$

$$J_k = \int_0^1 \frac{x \sin\left\{\omega \log \dfrac{1+x}{1-x}\right\}}{(\rho^2 - x^2)^{k-1/2}}\, dx \quad , \qquad \rho > 1 \,, \tag{2}$$

where $k = 0, 1, 2 \ldots\ldots\ldots\ldots\ldots\ldots$, ω is a constant, and

$$L_k = \int_a^b \frac{x^k \cos \omega\theta}{\{(x^2 - a^2)(b^2 - x^2)\}^{1/2}}\, dx \quad , \qquad k = 0\,,\, 2\,, \tag{3}$$

$$L_1 = \int_a^b \frac{x \sin \omega\theta}{\{(x^2 - a^2)(b^2 - x^2)\}^{1/2}}\, dx \quad , \tag{4}$$

where

$$\theta = \log\{(x - a)(b + x)/(x + a)(b - x)\}\,.$$

The evaluation of these integrals in closed form is, therefore of particular importance in fracture mechanics. In this note, we have evaluated these integrals in closed form for $\rho > 1$. These results have been used for obtaining expressions for the components of stress and stress intensity factors at the tip of the crack lying at the interface of two bonded dissimilar elastic solids.

SOME USEFUL RESULTS

In this section, we shall give some useful results for ready reference. These results will be used in the next sections for deriving closed form expressions for the integrals in question.

From Erdelyi's[6] we have

$$\int_0^\infty \operatorname{sech}^\nu \alpha x \cos xy \, dx = \frac{2^{\nu-2}}{\alpha \Gamma(\nu)} \Gamma\{(\alpha\nu + iy)/2\alpha\} \Gamma\{(\alpha\nu - iy)/2\alpha\} \tag{5}$$

and following relations for hypergeometric functions[7] –

$$c\,{}_2F_1(a, b-1; c; z)\, c\,{}_2F_1(a-1, b; c; z) = (b-a)\, z\, {}_2F_1(a, b; c+1; z) \tag{6}$$

$$(1-z)(b-a)\, {}_2F_1(a, b; c; z) = (c-a)\, {}_2F_1(a-1, b; c; z) - (c-b)\, {}_2F_1(a, b-1; c; z) \tag{7}$$

The following quadratic transformation will also be used in the next section[7]

$$ {}_2F_1(a, 1-a; c; -z) = (1+z)^{c-1} \left[(1+z)^{\frac{1}{2}} + z^{\frac{1}{2}} \right]^{2-2a-2c} .$$

$$\cdot\, {}_2F_1 \left[c+a-1,\, c-\tfrac{1}{2};\, 2c-1;\, 4z^{\frac{1}{2}} (1+z)^{\frac{1}{2}} \left\{ (1+z)^{\frac{1}{2}} + z^{\frac{1}{2}} \right\}^{-2} \right] \tag{8}$$

Lastly, we have the following Barne's contour integral representation

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)}\, {}_3F_2(a, b, c; d, e; z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c+s)\,\Gamma(-s)\,(-z)^s\, ds}{\Gamma(d+s)\,\Gamma(e+s)} \tag{9}$$

We further note that there is a general remark.[7] except for the connection between generalized hypergeometric series of argument z and z^{-1} which are the olution of the same differential equation, no linear transformation of ${}_{q+1}F_q$ seems to be known in the general case $q > 1$. For ${}_3F_2$ this relation is easily derived by evaluating the above integral by the calculus of residues as the sum of the residues at the poles $s = -a-k$, $s = -b-1$, $s = -c-m$ where k, 1, m=0, 1, 2, ... Thus we get

$$\begin{aligned} {}_3F_2(a, b, c; d, e, z) = {} & (-z)^{-a} \frac{\Gamma(b-a)\,\Gamma(c-a)\,\Gamma d\,\Gamma e}{\Gamma(d-a)\,\Gamma(e-a)\,\Gamma b\,\Gamma c}\, {}_3F_2 \left[a,\, \begin{matrix} 1+a-d,\, 1+a-e, \\ 1+a-b,\, 1+a-c, \end{matrix}\; z^{-1} \right] + \\ & + (-z)^{-b} \frac{\Gamma(a-b)\,\Gamma(c-b)\,\Gamma d\,\Gamma c}{\Gamma(d-b)\,\Gamma(e-b)\,\Gamma a\,\Gamma c}\, {}_3F_2 \left[b,\, \begin{matrix} 1+b-d,\, 1+b-e, \\ 1+b-a,\, 1+b-c, \end{matrix}\; z^{-1} \right] + \\ & + (-z)^{-c} \frac{\Gamma(a-c)\,\Gamma(b-c)\,\Gamma d\,\Gamma e}{\Gamma(d-c)\,\Gamma(e-c)\,\Gamma a\,\Gamma b}\, {}_3F_2 \left[c,\, \begin{matrix} 1+c-d,\, 1+c-e, \\ 1+c-a,\, 1+c-b, \end{matrix}\; z^{-1} \right] \end{aligned} \tag{10}$$

This formula is useful in determining the asymptotic behaviour of ${}_3F_2$ when z is very large.

EVALUATION OF INTEGRALS

In this section, we shall evaluate the following integrals which arise in the problems of fracture mechanics.

$$I_k = \int_0^1 \frac{\cos[\omega \log\{(1+x)/(1-x)\}]}{(\rho^2 - x^2)^{\frac{1}{k}-\frac{1}{2}}}\, dx, \quad \rho > 1 \tag{11}$$

$$J_k = \int_0^1 \frac{x \sin[\omega \log\{(1+x)/(1-x)\}]}{(\rho^2 - x^2)^{\frac{1}{k}-\frac{1}{2}}}\, dx \tag{12}$$

By substituting $x = \tanh \varphi$ and using the binomial expansion and (5) we get

$$I_k = \rho^{1-2k} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-)^s (k-\frac{1}{2})_n}{\rho^{2n}\, s!\, (n-s)!} \int_0^{\infty} \cos 2\omega\varphi \operatorname{sech}^{2s+2} \varphi \, d\varphi$$

$$= \rho^{1-2k} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-)^s (k-\frac{1}{2})_n\, 2^{2s}}{\rho^{2n}\, s!\, (n-s)!\, \Gamma(2s+2)} \Gamma(s+1+i\omega)\, \Gamma(s+1-i\omega)$$

$$= \frac{\Gamma\frac{1}{2}\, \Gamma(1+i\omega)\, \Gamma(1-i\omega)}{2\rho^{2k-1}} \sum_{s=0}^{\infty} \frac{(1+i\omega)_s\, (1-i\omega)_s\, (k-\frac{1}{2})_s}{s!\, \Gamma(s+1)\, (s+3/2)\, \rho^{2s}} \sum_{m=0}^{\infty} \frac{(k-\frac{1}{2}+s)_m}{m!\, \rho^{2m}}$$

$$= \frac{\Gamma(1+i\omega)\, \Gamma(1-i\omega)}{(\rho^2-1)^{k-\frac{1}{2}}} \sum_{s=0}^{\infty} \frac{(1+i\omega)_s\, (1-i\omega)_s (k-\frac{1}{2})_s}{s!\, (3/2)_s\, (1)_s} \left(-\frac{1}{\rho^2-1}\right)^s$$

$$= \frac{\pi\omega}{(\rho^2-1)^{k-\frac{1}{2}} \sinh \pi\omega}\, {}_3F_2 \left[1+i\omega,\, 1-i\omega,\, k-\tfrac{1}{2};\, 1,\, \tfrac{3}{2};\, -\frac{1}{\rho^2-1} \right], \quad \rho > \sqrt{2} \qquad (13)$$

Similarly, by substituting $x = \tanh \varphi$ in the integral (12) and by using the binomial expansion we have

$$J_k = \rho^{1-2k} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-)^s (k-\frac{1}{2})_n}{s!(n-s)!\, \rho^{2n}} \int_0^{\infty} \operatorname{sech}^{2s+2} \varphi \tanh \varphi \sin 2\omega\varphi \, d\varphi$$

Since

$$\int_0^{\infty} \operatorname{sech}^{2s+2} \varphi \tanh \varphi \sin 2\omega \varphi \, d\varphi = \frac{\omega}{s+1} \int_0^{\infty} \operatorname{sech}^{2s+2} \varphi \cos 2\omega\varphi \, d\varphi$$

we have

$$J_k = \omega \rho^{1-2k} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-)^s (k-\frac{1}{2})_n}{s!(n-s)!\, (s+1)\, \rho^{2n}} \int_0^{\infty} \operatorname{sech}^{2s+2} \varphi \cos 2\omega\varphi \, d\varphi$$

$$= \frac{\pi\omega^2}{(\rho^2-1)^{k-\frac{1}{2}} \sinh \pi\omega}\, {}_3F_2 \left[1+i\omega,\, 1-i\omega;\, k-\tfrac{1}{2};\, 2;\, 3/2;\, -\frac{1}{\rho^2-1} \right], \quad \rho > \sqrt{2}, \qquad (14)$$

The values of the above integrals for $\rho < \sqrt{2}$ can be calculated with the help of the formula (10). Next we evaluate the following integrals.

$$L_k = \int_a^b \frac{x^k \cos \omega\theta}{\{(x^2-a^2)(b^2-x^2)\}^{\frac{1}{2}}} \, dx, \qquad k = 0,\ 2 \qquad (15)$$

$$L_1 = \int_a^b \frac{x \sin \omega\theta}{\{(x^2-a^2)(b^2-x^2)\}^{\frac{1}{2}}} \, dx \tag{16}$$

where

$$\theta = \log \{(x-a)(b+x)/(x+a)(b-x)\}.$$

These integrals are evaluated by separating the real and imaginary parts of the following a integrals, which have been evaluated by making the substitution $x=a \cos^2 \varphi + b \sin^2\varphi$ and using the binomial expansion :

$$\int_a^b \frac{\exp(i\omega\theta)}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{\frac{1}{2}}} dx = \int_a^b (x-a)^{-\frac{1}{2}+i\omega} (x+a)^{-\frac{1}{2}-i\omega} (b-x)^{-\frac{1}{2}-i\omega} (b+x)^{-\frac{1}{2}+i\omega} dx \tag{17}$$

$$= \frac{\pi}{(a+b)\cosh \pi\omega} F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 1; z, -z)$$

$$\int_a^b \frac{x \exp(i\omega\theta)}{\{(x^2-a^2)(b^2-x^2)\}^{\frac{1}{2}}} dx = z\,\Gamma(\tfrac{1}{2}-i\omega)\,\Gamma(3/2+i\omega)\,F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega, 3/2; +i\omega, 2; z, -z) +$$

$$+ \frac{a}{a+b}\,\Gamma(\tfrac{1}{2}-i\omega)\,\Gamma(\tfrac{1}{2}+i\omega)\,F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 1, z, -z) \tag{18}$$

$$\int_a^b \frac{x^2 \exp(i\omega\theta)}{\{(x^2-a^2)(b-x^2)\}^{\frac{1}{2}}} dx = \frac{\pi a^2}{(a+b)\cosh \pi\omega} F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 1; z, -z) +$$

$$+ (b-a)\,\Gamma(\tfrac{1}{2}-i\omega)\,\Gamma(3/2+i\omega)\,F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 2; z, -z) -$$

$$- \Gamma\left(\frac{3}{2}+i\omega\right)\Gamma\left(\frac{3}{2}-i\omega\right)\frac{(b-a)z}{b} F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, 3/2-i\omega, 3/2+i\omega; 3; z, -z) \tag{19}$$

where $z=(b-a)/(b+a)$ and F_3 is hypergeometric function of two variables[7]. By separating the real and imaginary parts, we get

$$L_0 = \frac{\pi}{(a+b)\cosh \pi\omega}\ {}_2F_1(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; 1; z^2) \tag{20}$$

$$L_1 = \frac{z\,\pi\omega}{\cosh \pi\omega} F_3(\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega; 2; z, -z) \tag{21}$$

$$L_2 = \frac{\pi a^2}{(a+b)\cosh \pi\omega} F_3(\tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; 1; z, -z) +$$

$$+ \frac{\pi(b-a)}{2\cosh \pi\omega} F_3(\tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; 2; z, -z) +$$

$$+ \frac{(b-a)^2}{4(b+a)\cosh \pi\omega} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(\tfrac{1}{2}-i\omega)_p(\tfrac{1}{2}+i\omega)_q(\tfrac{1}{2}+i\omega)_p(\tfrac{1}{2}-i\omega)_q(2q+1)(p-q)\,z^p(-z)^q}{p!\,q!\,(3)_{p+q}} \tag{22}$$

It is worthwhile to remark that the above series are rapidly converging series. For any numerical computation of these integrals it is sufficient to consider only a few terms of the above series.

PARTICULAR CASES

In the next section we shall require the values of the integrals I_2 and J_2. From (13), with the help of (7) and (8) we have

$$I_2 = \frac{\pi\omega}{(\rho^2-1)^{3/2} \sinh \pi\omega} \, {}_2F_1\left(1+i\omega, 1-i\omega; 1; -\frac{1}{\rho^2-1}\right), \quad \rho > \sqrt{2} \tag{23}$$

$$= \frac{\pi\omega}{2\rho^2(\rho^2-1)^{1/2} \sinh \pi\omega}\left[F\left(i\omega, 1-i\omega; 1; -\frac{1}{\rho^2-1}\right) + F\left(-i\omega, 1+i\omega; 1; -\frac{1}{\rho^2-1}\right)\right], \quad \rho > \sqrt{2}$$

$$= \frac{\pi\omega}{2\rho^2(\rho^2-1)^{1/2} \sinh \pi\omega}\left[\left(\frac{\rho-1}{\rho+1}\right)^{i\omega} F\left(i\omega, \tfrac{1}{2}; 1; \frac{4\rho}{(\rho+1)^2}\right) + \left(\frac{\rho-1}{\rho+1}\right)^{-i\omega} F\left(-i\omega, \tfrac{1}{2}; 1; \frac{4\rho}{(\rho+1)^2}\right)\right], \quad \rho > 1, \tag{24}$$

Similarly, from (14) with the help of (6) and (8), we have

$$J_2 = \frac{\pi\omega^2}{(\rho^2-1)^{3/2} \sinh \pi\omega} F\left(1+i\omega, 1-i\omega; 2; \frac{-1}{\rho^2-1}\right), \quad \rho > \sqrt{2} \tag{25}$$

$$= \frac{\pi\omega}{2i(\rho^2-1)^{1/2} \sinh \pi\omega}\left[F\left(1+i\omega, -i\omega; 1; -\frac{1}{\rho^2-1}\right) - F\left(1-i\omega, i\omega; -\frac{1}{\rho^2-1}\right)\right], \quad \rho > \sqrt{2},$$

$$= \frac{\pi\omega}{2i(\rho^2-1)^{1/2} \sinh \pi\omega}\left[\left(\frac{\rho-1}{\rho+1}\right)^{-i\omega} F\left(-i\omega, \tfrac{1}{2}; 1; \frac{4\rho}{(\rho+1)^2}\right) - \left(\frac{\rho-1}{\rho+1}\right)^{i\omega} F\left(i\omega, \tfrac{1}{2}; 1; \frac{4\rho}{(\rho+1)^2}\right), \quad \rho > 1 \tag{26}$$

Next, we prove that I_1 is bounded.

$$|I_1| = \left| \int_0^1 \frac{\cos\left\{\omega \log \frac{(1+x)}{(1-x)}\right\}}{(\rho^2-x^2)^{1/2}} dx \right| \leqslant \int_0^1 \frac{dx}{\sqrt{\rho^2-x^2}} = \sin^{-1} \frac{1}{\rho}, \quad \rho > 1$$

The integral for I_1 is singular only for $\rho=1$, but this is a weak singularily of the integral. The value of this integral can easily be evaluated for $\rho=1$. For, on putting $x=\cos 2\theta$, we can easily obtain

$$\int_{-1}^{+1} \frac{\left(\frac{1+x}{1-x}\right)^{i\omega}}{\sqrt{1-x^2}} dx = \frac{\pi}{\cosh \pi\omega}$$

and on separating the real and imaginary parts, we get

$$\int_0^1 \frac{\cos\left(\omega \log \frac{1+x}{1-x}\right)}{\sqrt{1-x^2}} dx = \frac{\pi}{2\cosh \pi\omega}, \quad \int_0^1 \frac{\sin\left(\omega \log \frac{1+x}{1-x}\right)}{\sqrt{1-x^2}} dx = 0.$$

STRESS INTENSITY FACTOR

Lowengrub and Sneddon[1] considered the problem of determining the displacement and stress field in the vicinity of a penny-shaped crack situated at the interface of two half-spaces of different elastic materials bonded together along their plane boundary. We shall follow their notation. They have expressed normal stress component in the plane of the crack by the relation [1, 37]

$$\sigma_{zz}(\rho, 0+) = -\frac{K\beta}{\rho}\sqrt{\frac{2}{\pi}}\,\frac{\partial}{\partial\rho}\int_0^1 \frac{x\,S(x)}{(\rho^2 - x^2)^{\frac{1}{2}}}\,dx = K\beta\sqrt{\frac{2}{\pi}}\int_0^1 \frac{x\,S(x)}{(\rho^2 - x^2)^{3/2}}\,dx,\ \rho > 1, \quad (27)$$

Similarly the shearing stress in the plane of crack is given by the relation

$$\sigma_{\rho z}(\rho, 0+) = K\beta\sqrt{\frac{2}{\pi}}\,\frac{\partial}{\partial\rho}\int_0^1 \frac{R(x)}{(\rho^2 - x^2)^{\frac{1}{2}}}\,dx = -K\beta\rho\sqrt{\frac{2}{\pi}}\int_0^1 \frac{R(x)}{(\rho^2 - x^2)^{3/2}}\,dx,\ \rho > 1, \quad (28)$$

where

$$S(x) = \sqrt{\frac{2}{\pi}}\,\frac{f_0 \cosh\pi\omega}{\beta}\left[x\cos\omega\theta + \omega\sin\omega\theta\right],$$

$$R(x) = \sqrt{\frac{2}{\pi}}\,\frac{f_0 \cosh\pi\omega}{\beta}\left[x\sin\omega\theta - \omega\cos\omega\theta\right],$$

$$\theta = \log(1+x)/(1-x),\ \omega = \frac{1}{2\pi}\log\frac{k_1 G_2 + G_1}{k_2 G_1 + G_2} = \frac{1}{2\pi}\log\frac{\beta+\alpha}{\beta-\alpha} = \gamma\,.$$

The paper[1] contains a misprint. The value of $C_3 = -\gamma = -\omega$.

Substituting the value of $S(x)$ and $R(x)$ in (27) and (28) respectively, we get for $\rho > 1$

$$\sigma_{zz}(\rho, 0+) = \frac{2Kf_0\cosh\pi\omega}{\pi}\left[I_2\rho^2 + \omega J_2 - I_1\right] = \frac{2Kf_0\cosh\pi\omega}{\pi}\cdot\left[I_2\rho^2 + \omega J_2\right] + 0\ (1) \quad (29)$$

$$\sigma_{\rho z}(\rho, 0+) = \frac{-2K\rho f_0\cosh\pi\omega}{\pi}\left[J_2 - \omega I_2\right] \quad (30)$$

From (24) and (25) we have

$$I_2 = \frac{\pi\omega}{\rho^2(\rho^2-1)^{1/2}\sinh\pi\omega}\,(C X - S Y), \quad (31)$$

$$J_2 = -\frac{\pi\omega}{(\rho^2-1)^{1/2}\sinh\pi\omega}\,(C Y + S X), \quad (32)$$

where $X + i\,Y = {}_2F_1\left[i\,\omega, \tfrac{1}{2}; 1; \frac{4\rho}{(\rho+1)^2}\right], C + i\,S = \cos\omega\psi + i\sin\omega\psi,\ \psi = \log\frac{\rho-1}{\rho+1}$.

With the help of these equations we get

$$\sigma_{zz} + i\sigma_{\rho z} = \frac{Kf_0\,\omega\coth\pi\omega}{(\rho^2-1)^{1/2}}\left(1 + \frac{i\,\omega}{\rho}\right)\left[(X+iY)(1+\rho)(C+iS) - (\rho-1)(X-iY)(C-i\,S)\right] + O(1)$$

$$= \frac{Kf_0\,\omega\coth\pi\omega}{(\rho^2-1)^{1/2}}\left(1 + \frac{i\,\omega}{\rho}\right)(X+i\,Y)(1+\rho)\exp(i\,\omega\,\psi) + O(\sqrt{\rho-1}) \quad (33)$$

∴ The stress intensity factor is defined by the relation

$$N_1 + i\,N_2 = \lim_{\rho \to 1+}{}^{*}[\,(\rho - 1)^{\frac{1}{2}}\,(\sigma_{zz} + i\,\sigma_{\rho z})\,\exp(-i\,\omega\,\psi)\,]$$

Hence we get

$$N_1 + iN_2 = \frac{2K f_0}{\sqrt{2}}\;\omega \coth \pi\,\omega\;\Gamma(\tfrac{1}{2} - i\,\omega)\,(1 + i\,\omega)/\Gamma(1 - i\,\omega)\,\Gamma(\tfrac{1}{2})$$

$$= \frac{2K f_0}{\sqrt{2\pi}}\,\frac{\Gamma(2 + i\omega)}{\Gamma(\tfrac{1}{2} + i\,\omega)} \tag{34}$$

The above formula coincides with the expression for the stress intensity factor derived by Kassir and Bregmann[8]. The stresses in the crack plane in the vicinity of the rim of the crack are given by

$$\sigma_{zz}(\rho\,,\,0+) = \frac{1}{\sqrt{\rho - 1}}\,[\,N_1 \cos\omega\,\psi - N_2 \sin\omega\psi] + O(\sqrt{\rho - 1}) \tag{35}$$

$$\sigma_{\rho z}(\rho\,,0+) = \frac{1}{\sqrt{\rho - 1}}\,[N_1\,\sin\,\omega\psi\, + N_2 \cos\omega\psi\,] + O(\sqrt{\rho - 1}) \tag{36}$$

Lowengrub[5] has datermined the stress field in the vicinity of a pair of Griffith cracks located at the interface of the two bonded dissimilar elastic half planes. The stress-components in the plane of the cracks can be written as

$$\sigma_{yy}(x,\,0+) + i\,\sigma_{xy}(x,0+) = \frac{p_0}{\sqrt{(x^2 - a^2)(x^2 - b^2)}}\,[\,(x^2 + C'_2) - 2\,i\omega\,(b - a)\,x\,]\,e^{\omega\theta i} + \\ + O(1)\,,\, x > b \tag{37}$$

$$\sigma_{yy}(x,\,0+) + i\,\sigma_{xy}(x,\,0+) = -\frac{p_0}{\{\,(a^2 - x^2)(b^2 - x^2)\,\}^{1/2}}\,[\,(x^2 + C'_2) + 2\,i\omega\,(b - a)\,x\,]\,e^{+\omega\theta i} + \\ + O(1)\,,\; x < a \tag{38}$$

where

$$\theta = \log\{\,(x - a)(x + b)/(x + a)(x - b)\,\}.$$

where the two cracks have been defined by $a \leqslant |\,x\,| \leqslant b,\, y = 0,\; p_0$ is the constant pressure applied on the crack faces, ω is a known constant and the constant C_2' is given by the condition[5]

$$\int_a^b \{\,(x^2 - a^2)(b^2 - x^2)\,\}^{-1/2}\;[\,(x^2 + C'_2)\cos\omega\theta + 2\omega\,(b - a)\,x \sin\,\omega\theta]\,dx\; = 0 \tag{39}$$

This gives

$$C'_2 = -[\,L_2 + 2\,\omega\,(b - a)\,L_1\,]\,/\,L_0 \tag{40}$$

where L_0, L_1, L_2 are defined by (15) and (16). Lowengrub[5] has mentioned that the conatant C'_2 has to be calculated numerically. By virtue of the results (20) — (22), C'_2 can be expressed in closed form from which its numerical value can be easily computed.

If the normal and the shear stress intensity factors N_{1b} and N_{2b} at the edge $x = b$ and N_{1a} and N_{2a} at the edge $x = a$ are defined by the relations

$$N_{1b} + i\,N_{2b} = \lim_{x \to b+}\;[\,(x - b)^{1/2}\;(\sigma_{yy} + i\,\sigma_{xy})\,\exp(\,i\,\omega\,\theta\,)\,] \tag{41}$$

$$N_{1a} + i N_{2a} = \lim_{x \to a-} [(a - x)^{1/2} (\sigma_{yy} + i \sigma_{xy}) \exp (i a \omega b)] \quad (42)$$

Then we have

$$N_{1b} + i N_{2b} + \frac{p_0}{\{ 2 b (b^2 - a^2) \}^{1/2}} [(b^2 + C'_2) - 2 i \omega b (b - a)] \quad (43)$$

$$N_{1a} + i N_{2a} = - \frac{p_0}{\{ 2 a (b_2 - a_2) \}^{1/2}} [(a^2 + C'_2) + 2 i \omega a (b - a)] \quad (44)$$

REFERENCES

1. LOWENGRUB, M. & SNEDDON, I. N., *Int. J. Engg. Sci.*, 12 (1974), 387.
2. GLADWELL, G. M. L., *Int. J. Engg. Sci.*, 7 (1969), 295.
3. LOWENGRUB, M. & SNEDDON, I. N., *Int. J. Engg. Sci.*, 11 (1973), 1025.
4. LOWENGRUB, M. & SNEDDON, I. N., *Int. J. Engg. Sci.* 10 (1972), 289.
5. LOWENGRUB, M., *Int. J. Engg. Sci.*, 13 (1975), 731.
6. ERDELYI, A. et al, 'Tables of Integral Transforms', (Mc Graw Hill, New York) 1954.
7. ERDELYI, A., *et. al*, 'Higher tra scendental functions', (Mc Graw Hill; New York) 1954, p. 103, 112.
8. KASSIR, M. K. & BREGMANN, A. M., *J. Appl. Mech. Trans. ASME*, Ser. E. 39, (1972), 308.