MINIMUM BALLISTIC FACTOR PROBLEM OF SLENDER AXIAL SYMMETRIC MISSILES

V.B. TAWAKLEY & S.C. JAIN

Defence Science Laboratory, Delhi

(*Received* 23 *December*, 1977)

The problem of determining the geometry of slender, axisymmetric missiles of minimum ballistic factor in hypersonic flow has been solved via the calculus of variations under the assumptions that the flow is Newtonian and the surface-averaged skin-friction coefficient is constant. The study has been made for conditions of given length and diameter, given diameter and surface area, and given surface area and length. The earlier investigations[8] where only regular shapes were determined has been extended to cover those class of bodies which consist of regular shapes followed or preceded by zero slope shapes.

The problem of finding the missile shapes of minimum ballistic factor in hypersonic flow was previously treated by a number of authors, viz., Berman[1], Fink[2], Miele & Huang[3], Heidmann[4] and Tawakley & Jain[5,6&7]. In a recent paper Jain & Tawakley[8] gave a variational solution for extremising the sum of the products of the powers of several integrals and applied the same for finding missile geometries of minimum ballistic factor for the three cases when any two of the three geometric quantities of the missile viz., length l, diameter d and wetted area S are known in advance. Those class of shapes which are continuous and having positive slope everywhere were discussed and it was found that the solutions were valid upto certain critical values of the friction parameters $k_1 (\equiv 4c_f\, l^3/d^3)$, when l, d are known, $k_2 (\equiv 4c_f\, S^3/\pi^3 d^6)$, when S, d are known and $k_3 (\equiv 4c_f\, \pi^3 l^6/S^3)$, when S, l are known. In this paper the results have been extended to cover these cases where k_1, k_2 and k_3 exceed these upper limits. This involves considering those class of bodies which may have discontinuity in slope. In the l, d given case an analytical solution has been obtained instead of the numerical solution as proposed by Miele & Huang[3].

FORMULATION OF THE PROBLEM AND THE NECESSARY CONDITIONS FOR EXTREMAL SOLUTION

Under the assumptions that the flow is along the axis of the missile, pressure distribution obeys Newtonian law and the surface averaged skin-friction coefficient is constant, it was shown[8] that for finding the minimum ballistic factor, shape, the following three functional expressions have to be minimised.

$$C\,\frac{l^3}{d^3} = \frac{I_1}{I_3} + k_1\,\frac{I_2}{I_3} \qquad (l, d)\text{ given} \tag{1}$$

$$C\,\frac{S^3}{\pi^3 d^5} = \frac{I_1\, I_2^3}{I_3} + k_2\,\frac{I_2}{I_3} \qquad (S, d)\text{ given} \tag{2}$$

$$C\,\frac{\pi^2 l^5}{S^2} = \frac{I_1}{I_2^2\, I_3} + k_3\,\frac{I_2^2}{I_3} \qquad (S, l)\text{ given} \tag{3}$$

where

$$\left.\begin{aligned} I_1 &= \int_0^1 Y\, Y'^3\, dX \\ I_2 &= \int_0^1 Y\, dX \\ I_3 &= \int_0^1 Y^2\, dX \end{aligned}\right\} \tag{4}$$

$X\ (= x/l)$ and $Y\left(= \frac{2}{d}\, y\right)$ being dimensionless coordinates of the missile.

Now since we are considering the possibility of the missile having a zero-slope shape, i.e. $Y' \geqslant 0$, this inequality may be written as

$$Y' - Z^2 = 0 \tag{5}$$

where $Z(X)$ denotes a real variable.

According to the theory[8] the necessary condition for extremising functional expressions 1 to 3 is identical with extremising a new functional of the form

$$J = \int_0^1 F(X, Y, Y', Z, \mu_j, \nu)\, dX \qquad j = 1, 2, 3$$

where F denotes the fundamental function

$$F = \mu_1 Y Y' + \mu_2 Y + \mu_3 Y^2 - \nu (Y' - Z^2)$$

Here $\nu = \nu(X)$ is a variable Lagrange multiplier and μ_1, μ_2, μ_3 are constant multiplier determined[8] for the three cases as

$$\left.\begin{aligned} \mu_1 &= \frac{\lambda_1 \lambda_2}{\lambda_2 + k_1 \lambda_1} \\ \mu_2 &= \frac{k_1 \lambda_1 \lambda_2}{\lambda_2 + k_1 \lambda_1} \\ \mu_3 &= -\lambda_3 \end{aligned}\right\} (l,\ d) \text{ given} \qquad (6)$$

$$\left.\begin{aligned} \mu_1 &= \frac{\lambda_1}{1 + k_2 \lambda_1 \lambda^2_2} \\ \mu_2 &= \frac{3\lambda_2 + k_2 \lambda_1 \lambda^2_2}{1 + k_2 \lambda_1 \lambda^2_2} \\ \mu_3 &= -\lambda_3 \end{aligned}\right\} (S, d) \text{ given} \qquad (7)$$

$$\left.\begin{aligned} \mu_1 &= \frac{\lambda_1 \lambda^4_2}{\lambda^4_2 + k_3 \lambda_1} \\ \mu_2 &= \frac{2\lambda_2 (k_3 \lambda_1 - \lambda^4_2)}{\lambda^4_2 + k_3 \lambda_1} \\ \mu_3 &= -\lambda_3 \end{aligned}\right\} (S, l) \text{ given} \qquad (8)$$

where

$$\lambda_j = 1/I_j \qquad j = 1, 2, 3 \qquad (9)$$

From the calculus of variations it is known that the extremal solution must satisfy the Euler equations

$$6\,\mu_1 Y Y' Y'' + 2\,\mu_1 Y'^3 - \mu_2 - 2\mu_3 Y - \nu' = 0, \ \nu Z = 0 \qquad (10)$$

The second Euler equation admits the solution

$$\nu = 0 \text{ or } Z = 0$$

the first of which is called a regular shape and the second of which is called a zero slope shape. The extremal arc may be composed of one or both of these. Since the fundamental function F does not contain the independent variable X explicitly, the first Euler equation admits the first integral as

$$2\,\mu_1 Y Y'^3 - \mu_2 Y - \mu_3 Y^2 = c \qquad (11)$$

where c is a constant.

If the extremal arc is composed of more than one sub-arc then the corner conditions

$$\triangle(\mu_1 Y Y'^3) = \triangle(3\,\mu_1 Y Y'^3 - \nu) = 0$$

must be satisfied. Here $\triangle(\ldots\ldots.)$ denotes the difference between the quantities evaluated immediately after and before the corner point.

The first expression implies that the value of c does not change across the corner point. Also the two expressions admit the pair of solutions

$$Y = 0 \ , \quad \nu = 0 \ , \quad \triangle Y' \neq 0$$

and

$$Y \neq 0 \ , \quad \nu = 0 \ , \quad \triangle Y' = 0$$

These solutions imply that, (*i*) a discontinuity in slope can occur only on the axis of symmetry and (*ii*) regardless of whether there is a discontinuity in slope, the relation $\nu=0$ holds on both sides of the corner point.

For the extremal arc to be minimal the Weierstrass condition

$$E = \mu_1 Y \ (Y^{*\prime} + 2Y') (Y^{*\prime} - Y')^2 + \nu (Y^{*\prime} - Y') \geqslant 0$$

must be satisfied. Here unstarred symbols stand for the extremal arc and starred symbols for a comparison arc. For the regular shape, since μ_1 is positive for all the three cases [see eqns. (6), (7) & (8)], the positiveness of E is ensured as long as Y' and $Y^{*\prime}$. satisfy the constraint (5). For the zero slope shape, the positiveness of E is ensured provided $\nu \geqslant 0$ everywhere. Thus, we have

$Y' \geqslant 0$ for regular shape

$\nu \geqslant 0$ for zero-slope shape

$\nu = 0$ at the corner point

TOTALITY OF SOLUTIONS

Considering the zero slope shape $Y'=0$, i.e., $Y =$ const., the Euler equations (10) gives

$$\nu' = -(\mu_2 + 2\mu_3 Y)$$

which may be integrated as

$$\nu = -(\mu_2 + 2\mu_3 Y) (X - X_0) \tag{12}$$

where suffix 0 represents the corner point.

From (12) we observe that the transition from regular shape to zero slope shape and vice versa will occur when

$$c + \mu_2 Y_0 + \mu_3 Y_0^2 = 0 \tag{13}$$

Also (12) indicates that along the zero slope ν varies linearly with abscissa and so it can vanish at only one point of each zero slope shape and this is the corner point. This implies that the regular shape can be preceded or followed by no more than one zero slope shape and the equation of the zero slope shape can only be $Y = 0$ (a spike) and/or $Y = 1$ (cylinder).

If $Y = 0$ be the zero slope shape then from (12) & (13), we must have $\mu_2 > 0$ and $c = 0$.

If $Y = 1$ be the zero slope shape then again from (12) & (13), we must have $(\mu_2 + 2\mu_3) < 0$ and $c + \mu_2 + \mu_3 = 0$.

Since no more than two corner points and three sub-arcs can exist, the totality of extremal arcs consists of the following four classes of bodies.

Class *I* — Bodies composed of regular shape only ($\nu = 0$).

Class *II* — Bodies composed of a spike followed by a regular shape ($Y = 0 \rightarrow \nu = 0$).

Class *III* — Bodies composed of a regular shape followed by a cylinder ($\nu = 0 \rightarrow Y = 1$).

Class *IV* — Bodies composed of a spike followed by a regular shape followed by a cylinder ($Y = 0 \rightarrow \nu = 0 \rightarrow Y = 1$).

The most general form of the extremal arc is of class *IV* and with the help of (11) can be represented by the equations

$$Y = 0 \qquad 0 \leqslant X \leqslant X_{10}$$

$$\frac{X - X_{10}}{X_{20} - X_{10}} = \frac{\int_0^Y Y^{\frac{1}{2}} (c + \mu_2 Y + \mu_3 Y^2)^{-\frac{1}{2}} dY}{\int_0^1 Y^{\frac{1}{2}} (c + \mu_2 Y + \mu_3 Y^2)^{-\frac{1}{2}} dY} \qquad X_{10} \leqslant X \leqslant X_{20} \tag{14}$$

$$Y = 1 \qquad X_{20} \leqslant X \leqslant 1$$

where X_{10} and X_{20} represent the abscissae of the two possible transition (corner) points. Bodies of class *I* can be obtained from bodies of class *IV* by putting $X_{10} = 0$, $X_{20} = 1$. Similarly bodies of class *II* and class *III* can be obtained by putting $X_{20} = 1$ and $X_{10} = 0$ respectively. Thus from the above descussion we see that

$$X_{10} = 0 \,, \quad X_{20} = 1 \quad \text{for class } I \text{ bodies}$$
$$c = 0 \,, \quad X_{20} = 1 \quad \text{for class } II \text{ bodies}$$
$$X_{10} = 0 \,, \quad c + \mu_2 + \mu_3 = 0 \quad \text{for class } III \text{ bodies}$$
$$c = 0 \,, \quad \mu_2 + \mu_3 = 0 \quad \text{for class } IV \text{ bodies}$$

SOLUTION OF THE PROBLEM

From (4), (9) & (11), it can be deduced that

$$X_{20} \text{ or } (1 - X_{10}) = (2\mu_1)^{\frac{1}{2}} \int_0^1 Y^{\frac{1}{2}} (c + \mu_2 Y + \mu_3 Y^2)^{-\frac{1}{2}} dY \tag{15}$$

$$\frac{1}{\lambda_1} = (2 \mu_1)^{-\frac{3}{2}} \int_0^1 Y^{\frac{1}{2}} (c + \mu_2 Y_3 + \mu_3 Y^2)^{-\frac{3}{2}} dY \tag{16}$$

$$\frac{1}{\lambda_2} = (2 \mu_1)^{\frac{1}{2}} \int_0^1 Y^{4/3} (c + \mu_2 Y + \mu_3 Y^2)^{-\frac{1}{2}} dY + (1 - X_{20}) \tag{17}$$

$$\frac{1}{\lambda_3} = (2 \mu_1)^{\frac{1}{2}} \int_0^1 Y^{7/3} (c + \mu_2 Y + \mu_3 Y^2)^{-\frac{1}{2}} dY + (1 - X_{20}) \tag{18}$$

Combining (15), (17) & (18), we arrive at

$$c\,[X_{20} \text{ or } (1 - X_{10})] + \frac{3}{2} \frac{\mu_2}{\lambda_2} + \frac{2\mu_3}{\lambda_3} = \frac{3}{4} (2\mu_1)^{\frac{1}{2}} (c + \mu_2 + \mu_3)^{\frac{3}{2}} + $$
$$+ \left(\frac{3}{2} \mu_2 + 2\mu_3\right) (1 - X_{20}) \tag{19}$$

Jain & Tawakley[8] obtained bodies of class *I* only i.e., those bodies which consist of regular shape only. It was shown that such bodies can be obtained upto certain values of k_1, k and k_3 for (l, d) given, (S, d) given and (S, l) given cases respectively. Now we discuss those class of bodies which are minimal for values of k_1, k_2 and k_3 exceeding those limiting values.

Case 1 : Length and diameter are given

Jain & Tawakley[8] calculated that in this case

$$c = \frac{3\lambda_2}{\lambda_2 + k_1\lambda_1}$$

and so c is always positive and can never be zero. Therefore, the existence of bodies of class II and class IV are ruled out and the extremal bodies consist of class I and/or class III, i.e., the zero slope shape is $Y=1$ $(X_{10} = O)$ and so from (13), we have

$$c + \mu_2 + \mu_3 = 0 \tag{20}$$

Therefore (15), (17) & (18) reduce to

$$X_{20} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} \int_0^1 Y^{\frac{1}{3}} (1 - Y)^{-\frac{1}{3}} (1 + aY)^{-\frac{1}{3}} \, dY \tag{21}$$

$$\frac{1}{\lambda_2} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} \int_0^1 Y^{4/3} (1 - Y)^{-\frac{1}{3}} (1 + aY)^{-\frac{1}{3}} \, dY + (1 - X_{20}) \tag{22}$$

$$\frac{1}{\lambda_3} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} \int_0^1 Y^{7/3} (1 - Y)^{-\frac{1}{3}} (1 + aY)^{-\frac{1}{3}} \, dY + (1 - X_{20}) \tag{23}$$

where

$$a = \lambda_3/c \tag{24}$$

In the above, we make the substitution $Z=(1 - Y)$, we get

$$X_{20} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} (1 - \gamma)^{\frac{1}{3}} \int_0^1 Z^{-\frac{1}{3}} (1 - Z)^{\frac{1}{3}} (1 - \gamma Z)^{-\frac{1}{3}} \, dZ \tag{25}$$

$$\frac{1}{\lambda_2} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} (1 - \gamma)^{\frac{1}{3}} \int_0^1 Z^{-\frac{1}{3}} (1 - Z)^{4/3} (1 - \gamma Z)^{-\frac{1}{3}} \, dZ + (1 - X_{20}) \tag{26}$$

$$\frac{1}{\lambda_3} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} (1 - \gamma)^{\frac{1}{3}} \int_0^1 Z^{-\frac{1}{3}} (1 - Z)^{7/3} (1 - \gamma Z)^{-\frac{1}{3}} \, dZ + (1 - X_{20}) \tag{27}$$

where

$$\gamma = a/(1 + a) \tag{28}$$

These can be integrated in the form

$$X_{20} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} (1 - \gamma)^{\frac{1}{3}} \beta\left(\frac{2}{3}, \frac{4}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 2, \gamma\right) \tag{29}$$

$$\frac{1}{\lambda_2} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} (1 - \gamma)^{\frac{1}{3}} \beta\left(\frac{2}{3}, \frac{7}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right) + (1 - X_{20}) \tag{30}$$

$$\frac{1}{\lambda_3} = \left(\frac{2\mu_1}{c}\right)^{\frac{1}{3}} (1 - \gamma)^{\frac{1}{3}} \beta\left(\frac{2}{3}, \frac{10}{3}\right) F\, \tfrac{1}{3}, \tfrac{2}{3}, 4\, \gamma) + (1 - X_{20}) \tag{31}$$

In the above β and F represent Beta and Hypergeometric functions respectively. From (20), (24) & (28), we find that

$$\frac{\mu_2}{c} = \frac{1 - 2\gamma}{\gamma - 1} \tag{32}$$

But (6) & (19) give

$$\frac{\mu_2}{c} = \frac{k_1 \lambda_1}{3} \tag{33}$$

Combining (32) & (33), we get

$$\lambda_1 = \frac{3\,(2\gamma - 1)}{k_1\,(1 - \gamma)} \tag{34}$$

Again (6) & (34) give

$$\frac{2\mu_1}{e} = \tfrac{2}{3}\,\lambda_1 = \frac{2\,(2\gamma - 1)}{k_1\,(1 - \gamma)}$$

Thus (29), (30) & (31) can be written as

$$X_{20} = \left\{ \frac{2\,(2\gamma - 1)}{k_1} \right\}^{\frac{1}{3}} \beta\left(\frac{2}{3}, \frac{4}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 2, \gamma\right) \tag{35}$$

$$\frac{1}{\lambda_2} = \left\{ \frac{2\,(2\gamma - 1)}{k_1} \right\}^{\frac{1}{3}} \beta\left(\frac{2}{3}, \frac{7}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right) + (1 - X_{20}) \tag{36}$$

$$\frac{1}{\lambda_3} = \left\{ \frac{2\,(2\gamma - 1)}{k_1} \right\}^{\frac{1}{3}} \beta\left(\frac{2}{3}, \frac{10}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 4, \gamma\right) + (1 - X_{20}) \tag{37}$$

Further making use of (6), (20), (34), (36) & (37), we obtain

$$\left\{ \frac{54\,(2\gamma - 1)}{k_1} \right\}^{\frac{1}{3}} \left[\gamma\, \beta\left(\frac{2}{3}, \frac{10}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 4, \gamma\right) - (2\gamma - 1)\, \beta\left(\tfrac{2}{3}, \quad\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right) \right]$$
$$= (1 - \gamma)\,[1 - 3\,(1 - X_{20})] \tag{38}$$

Eliminating X_{20} from (35) & (38) and making use of the well-known properties of the Hypergeometric functions (see Appendix), we finally obtain

$$256\, k_1\,(1 - \gamma)^3 = 27\,(2\gamma - 1)\Big[2\,(1 - \gamma)\, \beta\left(\frac{2}{3}, \frac{4}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 2, \gamma\right) +$$
$$+ (2\gamma - 1)\, \beta\left(\frac{2}{3}, \frac{7}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right)\Big]^3 \tag{39}$$

This equation gives the values of γ for known values of k_1 ($>1 \cdot 6293$, see Appendix). Knowing γ we can obtain the transition point from (35). The relation between X_{20} and k_1 is presented in Fig. 1.

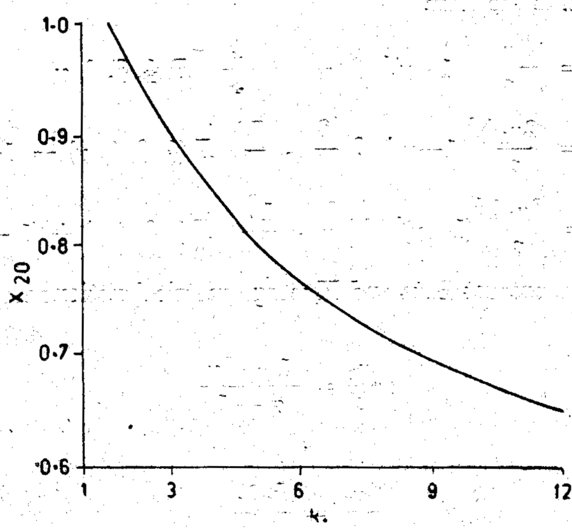

Fig. 1—Abscissa of the corner point, when (l, d_1) are given.

Knowing X_{20} the shape of the minimising curve of class *III* is known by using (14) as

$$\left.\begin{array}{ll} \dfrac{X}{X_{20}} = \dfrac{\int_0^Y Y^{\frac{1}{3}} (1-Y)^{-\frac{1}{3}} \left(1 + \frac{\gamma}{1-\gamma} Y\right)^{-\frac{1}{3}} dY}{\int_0^1 Y^{\frac{1}{3}} (1-Y)^{-\frac{1}{3}} \left(1 + \frac{\gamma}{1-\gamma} Y\right)^{-\frac{1}{3}} dY} & 0 \leqslant X \leqslant X_{20} \\ Y = 1 & X_{20} \leqslant X \leqslant 1 \end{array}\right\} \qquad (40)$$

This relation has been shown in Fig. 2. Further the values of λ_1, λ_2 and λ_3 can be calculated from (34) (36) & (37) respectively and then the value of the factor $C \frac{l^3}{d^3}$ is obtained from (1) and is represented in Fig. 3.

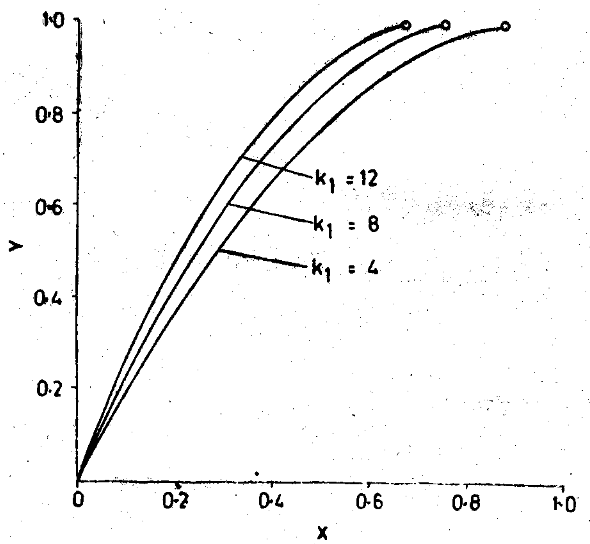

Fig. 2—Optimum shapes, when $(l,d,)$ are given.

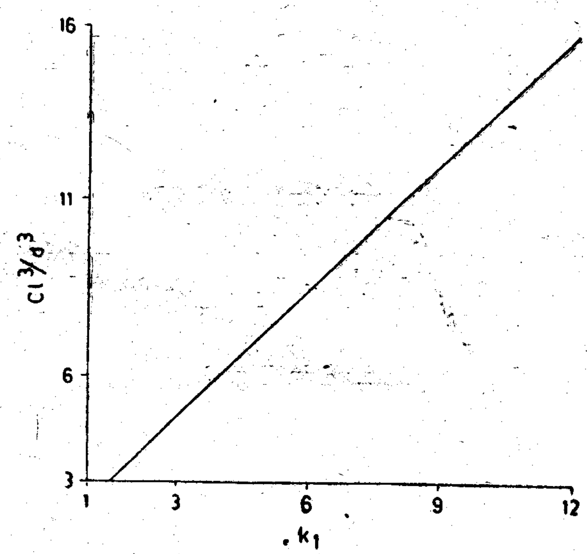

Fig. 3—Minimum value of the factor $C \frac{l^3}{d^3}$, when $(l.d.)$ are given.

Case 2 : Surface area and diameter are prescribed

Since the drag of a spike is zero regardless of its length and it does not also add to the volume, we can take any extremal arc of class *I* and generate from it an infinite number of equal ballistic factor solutions of class *II* by adding a spike of arbitrary length in front. Similarly, we can take any extremal arc of class *III* and generate from it an infinite number of equal ballistic factor solutions of class *IV* by adding a spike of arbitrary length in front. For these reasons, in case the length is free, only solutions of class *I* and/or class *III* are considered. These solutions occur when the friction parameter k_2 is smaller or larger than a critical value respectively. Jain & Tawakley[8] found that bodies of class *I* exist for $k_2 \leqslant 0{\cdot}9113$. Here we discuss bodies of class *III*. It was proved that when surface area and diameter are known, $c = 0$. Since $Y = 1$ is the zero slope shape, we have from (13).

$$\mu_2 + \mu_3 = 0 \qquad (41)$$

The shape of the minimising curve from (14) is easily represented as

$$\left.\begin{array}{ll} \dfrac{X}{X_{20}} = 1-(1-Y)^{\frac{2}{3}} & 0 \leqslant X \leqslant X_{20} \\ Y = 1 & X_{20} \quad X \leqslant 1 \end{array}\right\} \qquad (42)$$

Therefore (16), (17) & (18) give

$$\frac{1}{\lambda_1} = \frac{81}{160\, X_{20}^2} \tag{43}$$

$$\frac{1}{\lambda_2} = \left(1 - \frac{2}{5}\, X_{20}\right) \tag{44}$$

$$\frac{1}{\lambda_3} = \left(1 - \frac{11}{20}\, X_{20}\right) \tag{45}$$

But (15) gives

$$X_{20} = \frac{3}{2}\left(\frac{2\,\mu_1}{\lambda_3}\right)^{\frac{1}{3}} \tag{46}$$

Making use of (7), (43), (44) & (45), it can be deduced that

$$X_{20}\left[\,9\left(1 - \frac{2}{5}X_{20}\right)^2 + \frac{160}{27}K_2 X_{20}^{\,2}\right] = 40\left(1 - \frac{2}{5}\,X_{20}\right)^3 \tag{47}$$

This gives the transition point for known values of k_2 (> 0.9113) and is plotted in Fig. 4. Knowing X_0 we have the geometry of the minimal curve from (42) and is illustrated in Fig. 5. Also, λ_1, λ_2 and λ_3 are known from (43), (44) and (45) respectively and thus the values of $C\,S^3/\pi^3\,d^5$ are calculated from (2) and are represented in Fig. 6.

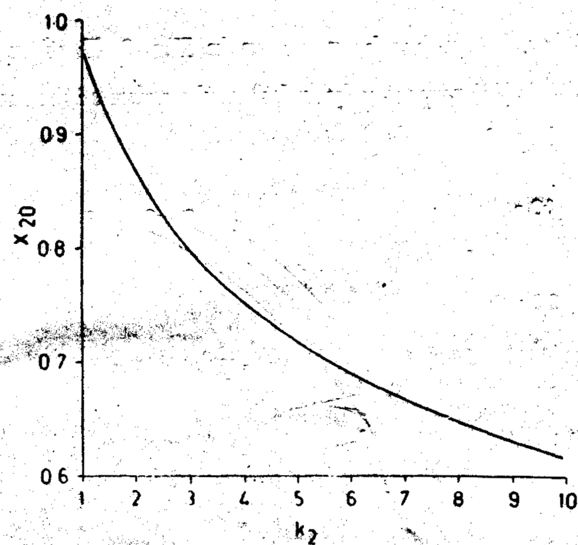

Fig. 4—Absoissa of the corner point, when (S,d) are given

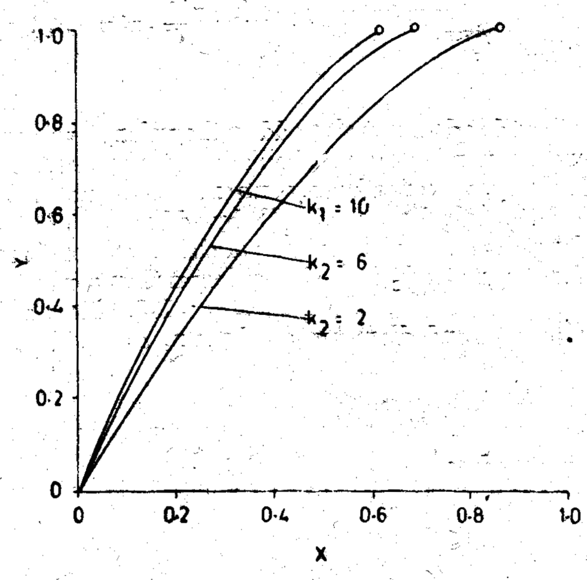

Fig. 5—Optimum shapes, when (S,d) are given.

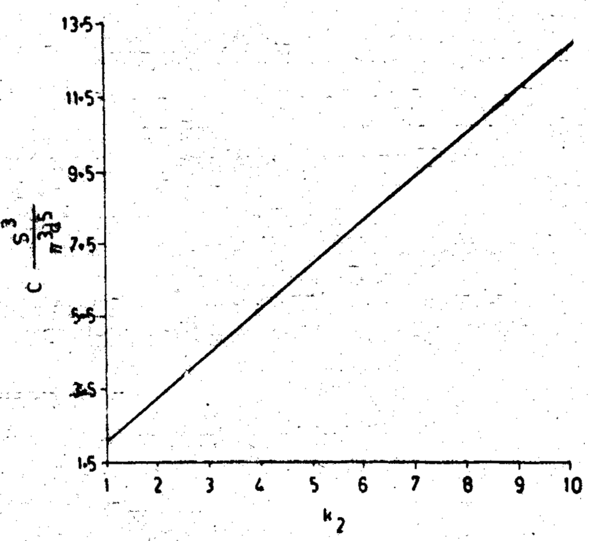

Fig. 6—Minimum value of the factor $C\,\frac{S^3}{\pi^3\,d^5}$, when(S,$d$) are given.

Case 3 : Surface area and length are known

Jain & Tawakley[8] deduced that

$$c = \frac{5\lambda_2^4 - k_3 \lambda_1}{\lambda_2^4 + k_3 \lambda_1} \tag{48}$$

In this case there is a possibility of c becoming zero. Jain & Tawakley[8] showed that for $c \geqslant 0$ bodies of class I exist and hold for $k_2 < 19 \cdot 5313$. For $c = 0$ the solution consists of bodies of class II ($X_{10} = 1$). Therefore, for zero slope shape, $Y = 0$. From (8) and (19), we see that $\mu_2 + \mu_3 = 0$ and so the shape of the optimizing curve of class II is

$$\left.\begin{array}{ll} Y = 0 & 0 \leqslant X \leqslant X_{10} \\ X = 1 - (1 - X_{10})(1 - Y)^{\frac{2}{3}} & X_{10} \leqslant X \leqslant 1 \end{array}\right\} \tag{49}$$

Using (49) in (16), (17) & (18) gives

$$\frac{1}{\lambda_1} = \frac{81}{160\,(1 - X_{10})^2} \tag{50}$$

$$\frac{1}{\lambda_2} = \frac{3\,(1 - X_{10})}{5} \tag{51}$$

$$\frac{1}{\lambda_3} = \frac{9\,(1 - X_{10})}{20} \tag{52}$$

Also from (15), we have

$$(1 - X_{10}) = \frac{3}{2}\left(\frac{2\,\mu_1}{\lambda_3}\right)^{\frac{1}{3}} \tag{53}$$

Making use of (8), (50), (51) & (52), lead to

$$X_{10} = 1 - \left(\frac{625}{32\,k_3}\right)^{\frac{1}{6}} \tag{54}$$

This gives the transition point as a function of k_3 ($>19 \cdot 5313$) and is represented in Fig. 7. Knowing X_{10}, we know from (49), the geometry of the minimising arc and is given in Fig. 8. Also λ_1, λ_2 and λ_3 are calculated from (50), (51) & (52) and so $C \dfrac{\pi^2 l^5}{S^2}$ is known from (3) and is represented in Fig. 9.

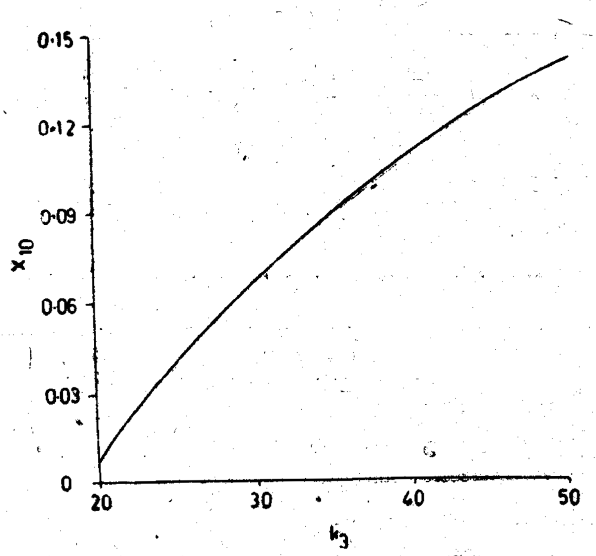

Fig. 7—Abscissa of the corner point, when (S, l) are given.

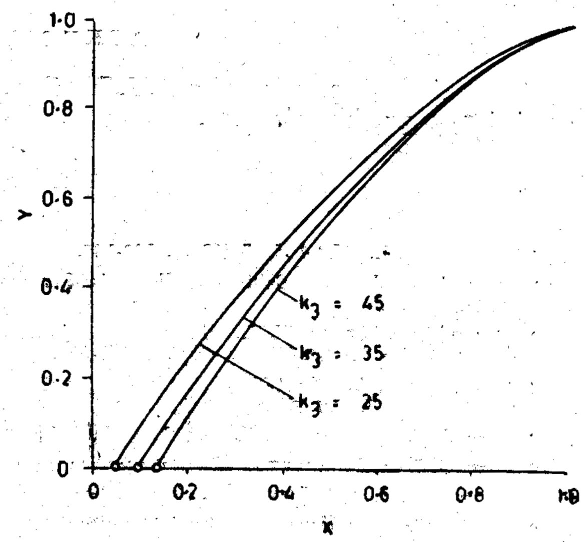

Fig. 8—Optimum shapes, when (S, l) are given.

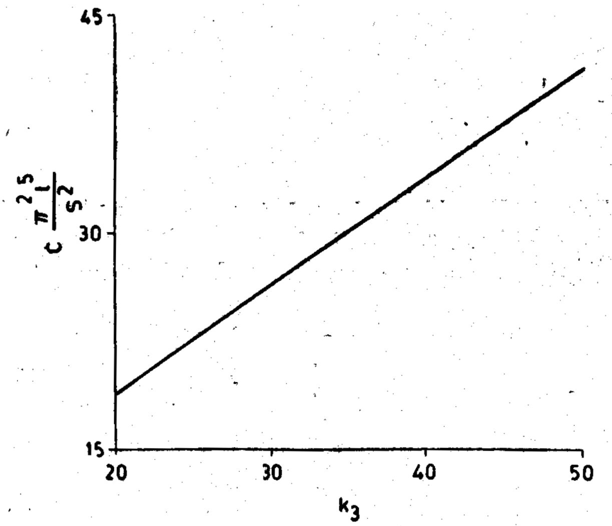

Fig. 9—Minimum value of the factor $C\,\frac{\pi^2 l^5}{S^2}$, when (S,l) are given.

APPENDIX

The analysis given for the first case when length and diameter are known in advance can be utilised to find analytically the critical value of k upto which only regular shapes of class I are possible. For this putting $X_{20} = 1$ in (35) & (38) give

$$1 = \left\{\frac{2\,(2\,\gamma - 1)}{k_1}\right\}^{\frac{1}{3}} \beta\left(\frac{2}{3},\, \frac{4}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 2, \gamma\right) \qquad (55)$$

$$3\,\gamma\,\beta\left(\frac{2}{3}, \frac{10}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 4, \gamma\right) - 3\,(2\gamma - 1)\,\beta\left(\frac{2}{3}, \frac{7}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right) = (1-\gamma)\left\{\frac{k_1}{2\,(2\gamma - 1)}\right\}^{\frac{1}{3}} \qquad (56)$$

Eliminating k_1 from these two expressions gives

$$3\gamma\;\beta\left(\frac{2}{3},\, \frac{10}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 4, \gamma\right) - 3\;(2\gamma - 1)\;\beta\left(\frac{2}{3},\, \frac{7}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right) =$$

$$= (1 - \gamma)\;\beta\left(\frac{2}{3},\, \frac{4}{3}\right) F\left(\tfrac{1}{3}, \tfrac{2}{3}, 2, \gamma\right)$$

Making use of the following well known property of the Hypergeometric functions

$$\gamma\,\{\gamma - 1 - (2\,\gamma - 1 - \alpha - \beta)\,x\}\; F(\alpha, \beta, \gamma, x) + (\gamma - \alpha)\,(\gamma - \beta)\,x\,F(\alpha, \beta, \gamma + 1, x) -$$

$$-\,\gamma\,(\gamma - 1)\,(1 - x)\,F(\alpha, \beta, \gamma - 1, x) = 0$$

the above may be simplified as

$$(1 - \gamma)\,F\left(\tfrac{1}{3}, \tfrac{2}{3}, 2, \gamma\right) - (2\,\gamma - 1)\,F\left(\tfrac{1}{3}, \tfrac{2}{3}, 3, \gamma\right) = 0$$

Solving this equation for γ the critical value of k_1 upto which regular shapes of class I are possible can be obtained by using (55) as 1·6293.

ACKNOWLEDGEMENT

The authors are grateful to Dr. A.K. Sreedhar, Director, Defence Science Laboratory, Delhi for granting permission to publish this work.

REFERENCES

1. BERMAN, R. J., *AIAA J*, **5** (1967), 166-167.
2. FINK, M. R., 'Maximum Ballistic Coefficient in Newtonian Flow' (United Aircraft Research Laboratories Report No. UAR-F-62), 1967.
3. MIELE, A. & HUANG, H. Y., 'Missile Shapes of Minimum Ballistic Factor' (Rice University Aero Astronautics Report 32), 1967 also *J. Opt. Theory Appl.* **1**, 151-164.
4. HEIDMANN, J. C., 'Blunt Nosed Missile Shapes of Minimum Ballistic Factor' (Rice University Aero Astronautics Report 32), 1967 also *J. Opt. Theory Appl.*, **1**, 1967, 242-252.
5. JAIN, S. C. & TAWAKLEY, V. B., *AIAA J.*, **8** (1970), 1711-1712.
6. JAIN, S. C. & TAWAKLEY, V. B., *Astronautica Acta*, **16** (1971), 277-279.
7. TAWAKLEY, V. B. & JAIN, S. C., *Astronautica Acta*, **18** (1973), 87-90.
8. JAIN, S. C. & TAWAKLEY, V. B., *Astronautica Acta* (Supplement), **18** (1974), 119-126.