# MINIMAL EXIT TRAJECTORIES WITH OPTIMUM CORRECTIONAL MANOEUVRES

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Minimal exit trajectories with optimum correctional manoeuvres to a rocket between two coplaner, non-coaxial elliptic orbits in an inverse square gravitational field have been investigated. Case of trajectories with no correctional manoeuvres has also been analysed. In the end minimal exit trajectories through specified orbital terminals are discussed and problem of ref. (2) is derived as a particular case.

*Nomenclature*

- $h_1$ twice the aerial velocity for launch orbit
- $h_2$ twice the aerial velocity for destination orbit
- $h$ twice the aerial velocity for transfer orbit
- $\alpha$ angle between radius vector of the launch point and major axis of the transfer trajectory
- $\beta$ angle between major axis of the launch and destination orbits.
- $\xi$ eccentricity of the transfer orbit
- $\xi_1$ eccentricity of the launch orbit
- $\xi_2$ eccentricity of the destination orbit
- $V$ launch velocity
- $V_1$ orbital velocity corresponding to launch orbit at the launch point
- $V_2$ orbital velocity corresponding to destination orbit at the destination point
- $\gamma$ launch angle i.e. angle between $V$ and local horizontal
- $\gamma_1$ angle between $V_1$ and local horizontal
- $\gamma_2$ angle between $V_2$ and local horizontal
- $\mu$ gravitational parameter

Lawden[1] has investigated an optimal correctional manoeuvre programme for a space vehicle by minimizing the mean characteristic velocity of the correctional manoeuvres $\overline{W}$. For similar launch operation procedure and same set of detection and correction instruments with some pre-assigned acceptable value to $S$ for all ballistic transfer trajectories between two specified launch and destination orbits, it can be assumed that the total transfer time $\tau_0$ will serve as a measure of corresponding $\overline{W}_{min}$ along the trajectory and $\overline{W}_{min}$ is given by[2]

$$\overline{W}_{min} \doteq ke \log \left( \frac{m}{k} \cdot \frac{\tau_0}{S} \right) \qquad \text{(A)}$$

If $m > ke$

$$\overline{W}_{min} \doteq m + ke \log \left( \frac{\tau_0}{S} \right) \qquad \text{(B)}$$

where $m$ is the statistical mean value of the error in the impulsive velocity change brought at the time of launch, $k$ the statistical mean value of the subsequent correctional manoeuvres, $\tau_0$ the total time of flight of the vehicle from the given launch to the destination point, $S$ (a pre-assigned value) represents the time interval between final correctional manoeuvre and arrival of the vehicle at the target and $e$ is the base of the natural logarithms. Prior to launching, $m$ and $k$ are assumed to have been determined.

The object of the present paper is to investigate the minimal trajectories between coplaner, non-coaxial elliptic orbits in an inverse square gravitational field. The optimization criterion adopted is the least propellant energy consumption during the impulsive launch and correctional manoeuvre imparted to the rocket. Minimal exit trajectories with no correctional manoeuvres have been investigated. Minimal exit trajectories through specified terminal points on the orbits are also analysed and problem of ref. (2) is shown to be a particular case of the present problem.

## TRANSFER TIME AND PARAMETERS

Let the equations to launch, transfer and destination orbits be

$$h_1^2 = \mu r (1 + \xi_1 \cos \theta) \quad (1)$$
$$h^2 = \mu r [1 + \xi \cos (\theta - \theta_1 + \alpha)] \quad (2)$$
$$h_2^2 = \mu r [1 + \xi_2 \cos (\theta - \beta)] \quad (3)$$

The impulsive velocity change $\triangle V$ of the rocket at the launch point $(r_1, \theta_1)$ on the launch orbit will be

$$\triangle V = [V^2 + V_1^2 - 2V V_1 \cos (\gamma - \gamma_1)]^{1/2} \quad (4)$$

If $(r_2, \theta_2)$ be the entry point of the rocket on the destination orbit, the relationship between $V$ and $\gamma$ can be expressed as[3]

$$\frac{r_1}{r_2} = \frac{\mu}{r_1 V^2} \left( \frac{1 - \cos \lambda}{\cos^2 \gamma} \right) + \frac{\cos (\gamma + \lambda)}{\cos \gamma} \quad |\gamma| \leqslant \pi/2 \quad (5)$$

where

$$\theta_2 = \theta_1 + \lambda$$

The transfer time $F$ of the rocket along the transfer trajectory is given by

$$F = \sqrt{\frac{a^3}{\mu}} \left[ (\psi_2 - \psi_1) - \xi (\sin \psi_2 - \sin \psi_1) \right] \quad (6)$$

where $\psi_1$ and $\psi_2$ are the eccentric anomalies of the exit and entry points on the transfer trajectory and

$$\cos \psi = \frac{\xi + \cos (\theta - \theta_1 + \alpha)}{1 + \xi \cos (\theta - \theta_1 + \alpha)} \quad (7)$$

From orbital dynamics, it can be shown that

$$\alpha = \tan^{-1} \left[ \frac{V^2 \sin \gamma \cos \gamma}{(V \cos \gamma)^2 - u_1 \mu} \right] \quad (8)$$

$$\xi = \left[ \left( \frac{V^2}{u_1 \mu} - 1 \right)^2 \cos^2 \gamma + \sin^2 \gamma \right] \quad (9)$$

and

$$a = \left( 2u_1 - \frac{V^2}{\mu} \right)^{-1} \quad (10)$$

where $a$ is semi-length of the major axis of the transfer trajectory and $u_1 = 1/r_1$.

Equation (5) can be written as

$$V^2 = \frac{\mu (1 - \cos \lambda) \sec^2 \gamma}{r_1 (r_1/r_2 + \sin \lambda \tan \gamma - \cos \lambda)} \quad (11)$$

Also for launch orbit, we have

$$\tan \gamma_1 = \frac{\mu \xi_1 \sin \theta_1}{u_1 h_1^2} \quad (12)$$

$$V_1 = \sqrt{\mu} \left[ 2 u_1 - \frac{\mu (1 - \xi_1^2)}{h_1^2} \right]^{1/2} \quad (13)$$

Eq. (6) can be transformed into

$$F = \frac{a^{3/2}}{\sqrt{\mu}} \left[ 2 \tan^{-1} \left( \frac{\sqrt{1 - \xi^2} \tan \lambda/2}{1 + \xi (\cos \alpha - \sin \alpha \tan \lambda/2)} \right) - \xi (1 - \xi^2)^{1/2} \left\{ \frac{\sin \lambda (\xi + \cos \alpha) - \sin \alpha (1 - \cos \lambda)}{(1 + \xi \cos (\lambda + \alpha))(1 + \xi \cos \alpha)} \right\} \right] \quad (14)$$

It can be seen that $F$ and $\triangle V$ are functions of three independents variables $\theta_1$, $\lambda$ and $\gamma$ by virtue of Eqs. (8—11) and (11—13) respectively.

## MINIMAL EXIT TRANSFER

For optimal transfer between specified orbits characterized by minimum energy consumption in exit and correctional manoeuvres ($\triangle W_{min} + \triangle V$) should be minimum and hence from (A) and (B), we have

$$\frac{\partial F}{\partial \theta_1} + \frac{F}{ke} \cdot \frac{\partial (\triangle V)}{\partial \theta_1} = 0 \quad (15)$$

$$\frac{\partial F}{\partial \lambda} + \frac{F}{ke} \cdot \frac{\partial (\triangle V)}{\partial \lambda} = 0 \quad (16)$$

$$\frac{\partial F}{\partial \gamma} + \frac{F}{ke} \cdot \frac{\partial (\triangle V)}{\partial \gamma} = 0 \quad (17)$$

From Eq. (14), we get

$$\frac{\partial F}{\partial \theta_1} = \frac{a^{3/2}}{\sqrt{\mu}} \left( L X + \frac{T r_1 \tan \gamma}{2 - G} \right) \quad (18)$$

$$\frac{\partial F}{\partial \lambda} = \frac{a^{3/2}}{\sqrt{\mu}} \left[ MX + \frac{(1 - \xi^2)^{3/2}}{(1 + \xi \cos (\lambda + \alpha))^2} \right] \quad (19)$$

$$\frac{\partial F}{\partial \gamma} = \frac{a^{3/2}}{\sqrt{\mu}} \left[ \frac{N\,T\,r_1}{(2-G)^2} + \frac{SG^2 \cos\gamma}{\xi} \left( \sin\gamma + (1-G)\, \frac{\sin\lambda \cos\gamma}{1-\cos\lambda} \right) + \right.$$

$$\left. + \frac{K(r_2 - r_1)\,(1-\cos\lambda)}{Hr_2 \sin\gamma \cos\gamma} \right] \tag{20}$$

where

$$G = r_1 V^2/\mu$$

$$H = (1 - r_1/r_2)^2 \cot\gamma - 2\,(1 - r_1/r_2) \sin\lambda + 2 \tan\gamma\,(1-\cos\lambda)$$

$$S = \sqrt{1-\xi^2} \left[ \frac{\sin\alpha\,(2+\xi\cos\alpha)}{(1+\xi\cos\alpha)^2} - \frac{\sin(\alpha+\lambda)\,(2+\xi\cos(\alpha+\lambda))}{(1+\xi\cos(\alpha+\lambda))^2} \right]$$

$$T = \frac{3\sqrt{\mu}\,(2-G)\,F}{2r_1\, a^{3/2}}$$

$$K = (1-\xi^2)^{3/2} \left[ \frac{1}{(1+\xi\cos(\lambda+\alpha))^2} - \frac{1}{(1+\xi\cos\alpha)^2} \right]$$

$$L = \frac{r_1\, G^2 \cos^2\gamma\,(\tan\gamma_2 - \tan\gamma_1)}{r_2\,(1-\cos\lambda)}$$

$$M = \frac{G^2\cos^2\gamma}{(1-\cos\lambda)^2} \left[ (r_1/r_2 - 1)\sin\lambda + (1-\cos\lambda)\left(\tan\gamma + \frac{r_1}{r_2}\tan\gamma_2\right) \right]$$

$$N = G \left[ 2\tan\gamma - \frac{G\sin\lambda}{1-\cos\lambda} \right]$$

and

$$X = \frac{S(G-1)\cos^2\gamma}{\xi} - \frac{K\,(1-\cos\lambda)^2}{G^2\,H\cos^2\gamma} + \frac{Tr_1}{(2-G)^2}$$

Also from Eq. (4), we have

$$\frac{\partial(\triangle V)}{\partial\theta_1} = \frac{1}{2}\,(\triangle V)^{-\frac{1}{2}} \left[ \left( \frac{\mu L}{r_1} - V^2\tan\gamma_1 \right) \left( 1 - \frac{V_1\cos(\gamma-\gamma_1)}{V} \right) - \right.$$

$$\left. -2 \left\{ \frac{\mu\sin\gamma_1}{h_1} \left( V_1 - V\cos(\gamma-\gamma_1) \right) + V\sin(\gamma-\gamma_1) \left( V_1 - \frac{\mu}{r_1 V_1} \right) \right\} \right] \tag{21}$$

$$\frac{\partial(\triangle V)}{\partial\lambda} = \frac{1}{2}\,(\triangle V)^{-\frac{1}{2}} \left[ \frac{\mu M}{r_1} \left( 1 - \frac{V_1\cos(\gamma - {}_1)}{V} \right) \right] \tag{22}$$

$$\frac{\partial(\triangle V)}{\partial\gamma} = \frac{1}{2}\,(\triangle V)^{-\frac{1}{2}} \left[ \frac{\mu N}{r_1} \left( 1 - \frac{V_1\cos(\gamma-\gamma_1)}{V} \right) + 2\,VV_1\sin(\gamma-\gamma_1) \right] \tag{23}$$

Substitution from Eqs. (18) to (23) in (15) to (17) gives three equations in unknowns $\theta_1$, $\lambda$ and $\gamma$ which can be numerically solved for known values of launch and destination orbital parameters. Having known $\theta_1$, $\lambda$ and $\gamma$ remaining elements like launch velocity, entry point, eccentricity of the minimal exit trajectory can be determined.

*Case of No Correctional Manoeuvres*

In case of exit trajectory with no correctional manoeuvres Eqs. (15) to (17) reduce to

$$\frac{\partial(\triangle V)}{\partial\theta_1} = 0 \tag{24}$$

$$\frac{\partial(\triangle V)}{\partial\lambda} = 0 \tag{25}$$

$$\frac{\partial (\triangle V)}{\partial \gamma} = 0 \qquad (26)$$

Substitutions from Eq. (4) in (24) to (26), differentiation and simplification gives

$$Y \left( \frac{\mu L}{r_1} - V^2 \tan \gamma_1 \right) - 2 \left\{ \frac{\mu \sin \gamma_1}{h_1} \left( V_1 - V \cos (\gamma - \gamma_1) \right) + \right.$$

$$\left. + V \sin (\gamma - \gamma_1) \left( V_1 - \frac{\mu}{r_1 V_1} \right) \right\} = 0 \qquad (27)$$

$$\mu M Y = 0 \qquad (28)$$

and

$$\mu N Y + 2 r_1 V V_1 \sin (\gamma - \gamma_1) = 0 \qquad (29)$$

where

$$Y = 1 - \frac{V_1 \cos (\gamma - \gamma_1)}{V}$$

Eqs. (28) and (29) when solved numerically gives $\theta_1$, $\lambda$ and $\gamma$ for minimal exit trajectory with no correctional manoeuvres and then other elements of the trajectory can be known.

*Minimal Exit Trajectories for Specified Orbital Terminals*

Minimal exit trajectories with correctional manoeuvers for specified orbital terminals can be classified in three groups.

(*i*) when terminal on launch orbit is specified,

(*ii*) when terminal on destination orbit is specified,

(*iii*) when terminals on both the launch and destination orbits are specified.

For trajectories of group (*i*) since launch point ($r_1$, $\theta_1$) is specified, Eq. (15) vanishes and $F$ and $\triangle V$ reduce to functions of two variables $\lambda$ and $\gamma$. The optimality equations in this case will be given by Eqs. (16) and (17). Solutions of Eqs. (16) and (17) will determine the values of $\lambda$ and $\gamma$ for the minimal exit trajectory and thereafter the other parameters of the trajectory can be known proceeding as before.

For group (*ii*) when destination point ($r_2$, $\theta_2$) is specified

$$\theta_2 = \theta_1 + \lambda = \text{const} \qquad (30)$$

Substitution for $\lambda$ from Eq. (30) in (4) and (14) transforms $\triangle V$ and $F$ as function of $\theta_1$ and $\gamma$ and optimality equations are given by Eqs. (15) and (17) which can be solved as in the previous case. In case (*iii*) both the terminals ($r_1$, $\theta_1$) and ($r_2$, $\theta_2$) are specified, hence Eqs. (15) and (16) vanish and thus the optimality equations will be Eq. (17) as obtained in ref. (2).

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