FLOW OF A CONDUCTING FLUID WITH SUSPENSION OF PARTICLES IN CYLINDERS WITH ARBITRARY TIME VARYING PRESSURE GRADIENT

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The problems of unsteady flow of a viscous incompressible conducting fluid containing a dilute suspension of small inert spherical particles in cylinders with circular and sectorial cross-sections, in the presence of a radial magnetic field, have been discussed. The analysis applies to flows with pressure gradients which are arbitrary functions of time. Explicit expressions for the exact velocities of the fluid and that of the particles are obtained by using the methods of operational calculus—the Laplace transform, the finite Hankel transforms etc. Numerical results have been obtained for the developing flow due to a constant pressure gradient, when the flow is taking place in a coaxial circular cylinder. The effects of mass concentration and relaxation time of particles and the magnetic field on the velocities have also been discussed.

The study of MHD flow problem is important because the behaviour of the conducting fluid is considerably changed in the presence of magnetic field and many new phenomena are observed. Gold[1] has obtained an exact solution of steady one-dimensional flow of an incompressible viscous, electrically conducting fluid through a circular cylinder in the presence of an applied (transverse) uniform magnetic field. Singh & Rizvi[2] have investigated the impulsive motion of a viscous liquid contained between two concentric circular cylinders in the presence of a radial magnetic field $[(H/r)\vec{i}]$. Singh[3] has also discussed the same problem in the presence of an axial magnetic field H.

In recent years, the problems of fluid flow embedded with particles have gained increased attention of mathematicians and engineers in view of their applications in a wide variety of engineering situations including environmental pollution, combustion fluidization, and more recently blood flow etc. It is due to this reason a number of studies of flow of a fluid embedded with particles[4–14] have appeared in literature. Rao[10] has discussed the flow of a dusty gas in a circular cylinder under the influence of an exponentially time varying pressure gradient. Singh & Pathak[11] have considered the flow of the dusty gas in a tube with sector of a circle as cross-section under the influence of exponential pressure gradient, Gupta & Gupta[12–13] have studied the flow of the dusty gas under the influence of an arbitrary time varying pressure gradient in a rectangular channel in circular cylinder and in a cylinder whose cross-section is a sector of a circle respectively.

The problems of flow of fluid embedded with particles in the presence of magnetic field become more complex but not intractable, Yang & Healy[14] have studied the flow induced in an incompressible fluid embedded with particles by an infinite flat plate set into motion in its plane by oscillation and impulse in the presence of a transverse magnetic field, neglecting the electromagnetic induced effect.

The present paper is a generalization of the work of Gupta & Gupta[13]. In this paper, the authors have studied some of the problems of flow of a viscous electrically conducting fluid containing a dilute suspension of small inert spherical particles in cylinders with non-conducting walls, in the presence of a radial magnetic field. The flow has been considered in (*i*) a circular cylinder, (*ii*) an annulus bounded by two coaxial circular cylinders, (*iii*) a cylinder whose cross-section is a sector of a circle and (*iv*) a cylinder whose cross-section is an annular sector. It is the purpose of this paper to develop the general time dependent flow model and to obtain explicit

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EQUATIONS OF MOTION

expressions for both the fluid and particles velocities, when the pressure gradient is an arbitrary function of time, in exact form. The changes in velocity profiles of the fluid and particles with time, for a constant pressure gradient, have been drawn graphically when the flow is taking place through the annulus.

In the present investigation, it is assumed that the cylinders are of infinite length and the flow is considered along z-axis, which concides with the axis of the cylinder. The appropriate equations, which assume Stokes drag law, are the well-known momentum equations for the fluid and particles[4]. These equations after introducing the electromagnetic force, in cylindrical polar coordinates (R, θ, z) are

$$\frac{\partial \vec{u}}{\partial T} = -\frac{1}{\rho} \nabla \vec{p} + \nu \nabla^2 \vec{u} + \frac{KN_0}{\rho} (\vec{v} - \vec{u}) + \frac{1}{\rho} (\vec{J} \times \vec{B}) \quad (1)$$

$$\frac{\partial \vec{v}}{\partial T} = \frac{K}{m} (\vec{u} - \vec{v}),$$

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \right), \quad (2)$$

where $\vec{u}$ and $\vec{v}$ are velocity vectors of the fluid and particles respectively, m is the mass of a particle, N_0 the number density of the particles which is constant throughout the motion, K the Stokes resistance coefficient, T the time, p the pressure, ν the kinematic viscosity, ρ the density of the fluid, $\vec{J}$ and $\vec{B}$ are given by Mexwell's equations and Ohm's law, namely

$$\text{curl } \vec{H} = 4 \pi \vec{J} \quad (3)$$

$$\text{div } \vec{B} = 0 \quad (4)$$

$$\text{curl } \vec{E} = 0 \quad (5)$$

$$\vec{J} = \sigma_e [\vec{E} + (\vec{u} \times \vec{B})] \quad (6)$$

In the present investigation, it is assumed that the effects of the induced magnetic field and electric field produced by the motion of the electrically conducting fluid are negligible and no external force field is applied. With these assumptions the magnetic term $\vec{J} \times \vec{B}$ in Eq. (1) is given by

$$\vec{J} \times \vec{B} = -\frac{B_0^2 \sigma_e}{R^2} \vec{u} \quad (7)$$

where B_0 ($= \mu_e H_0$) is electro-magnetic induction, μ_e and σ_e are the magnetic permeability and the conductivity of the fluid respectively.

In the present investigation the velocity components of the fluid in the radial, tangential and axial are direction

$u_R = 0, \ u_\theta = 0, \ u_z = u_z (R, \theta, t)$
and those for the particles are

$v_R = 0, \ v_\theta = 0, \ v_z = v_z (r, \theta, t).$

Introducing the following non-dimensional quantities

$u = a u_z/\nu, \ v = a v_z/\nu, \ z' = z/a, \ r = R/a$

$\rho' = \rho a^2/\rho\nu^2, \ t = T\nu/a^2$, the equations (1) and (2) in view of (7) become

$$\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial z'} + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) + \beta (v - u) - \frac{M^2}{r^2} u \quad (8)$$

and

$$\frac{\partial v}{\partial t} = \tau' (u - v), \quad (9)$$

where $f = N_0 m/\rho$ is the mass concentration of particles, $\tau = (m/K)/(a^2/\nu)$ is the dimensionless relaxation time of particles, $\beta = f/\tau = N_0 K a^2/\rho\nu$, $\tau' = 1/\tau$ and $M = (\sigma_e B_0^2/\mu)^{\frac{1}{2}}$ is the Hartmann number.

Using the above fundamental equations the authors have solved some basic problems of axially symmetric motion in the following sections.

SOLUTION OF THE PROBLEMS

(i) Circular Cylinder

Let us consider the motion of the conducting fluid in a circular cylinder of radius a. For the present geometry, due to axi-symmetry the velocities are independent of θ. Therefore, the Eqs. (8) and (9) governing the motion reduced to

$$\frac{\partial u}{\partial t} = ft + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + \beta (v - u) - \frac{M^2}{r^2} u \quad (10)$$

and

$$\frac{\partial v}{\partial t} = \tau' (u - v), \quad (11)$$

where $\varrho p'/\partial z' = -f(t)$, is an arbitrary function of time t.

To solve the problem, we have coupled the Eqs. (10) and (11) with the help of Laplace transform. Therefore, it is necessary to specify boundary conditions only for the fluid, whereas the initial conditions are required for the particles as well as for the fluid.

The initial and boundary conditions for the problem are

Initial conditions :

$$\left.\begin{aligned} u(r, 0) &= 0 \\ v(r, 0) &= 0 \end{aligned}\right\} \quad (12)$$

Boundary conditions :

$$\left.\begin{aligned} u(1, t) &= 0 \\ u(0, t) &= \text{finite} \end{aligned}\right\} \quad (13)$$

Applying the Laplace transform to Eqs. (10) and (11) under the conditions (12) and then solving the transformed equations for $\bar{u}$ and $\bar{v}$, we get

$$s\bar{u} = \bar{f}(s) + \left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{M^2}{r^2}\bar{u}\right) - \frac{\beta s}{\tau' + s}\bar{u}, \quad (14)$$

$$\bar{v} = \frac{\tau'}{\tau' + s}\bar{u}, \quad (15)$$

where $\bar{u}$, $\bar{v}$ and $\bar{f}(s)$ are the Laplace transforms of the respective expressions and s is the parameter of the transformation. The boundary conditions (13) transformed to

$$\left.\begin{aligned} \bar{u}(1, s) &= 0 \\ \bar{u}(0, s) &= \text{finite} \end{aligned}\right\} \quad (16)$$

Now applying the finite Hankel transform defined by

$$\bar{\bar{u}}(p_i, s) = \int_0^1 \bar{u}(r, s)\, r\, J_M(p_i r)\, dr, \quad (17)$$

where p_i is a positive root of the equation

$$J_M(p_i) = 0, \quad (18)$$

to Eq. (14) and on using the transformed boundary conditions (16),

we get

$$s\bar{\bar{u}} = -\frac{\bar{f}(s)}{p_i} J_{M-1}(p_i)\, S_{1,M}(p_i) - p_i^2 \bar{\bar{u}} - \frac{\beta s}{\tau' + s}\bar{\bar{u}}, \quad (19)$$

where

$S_{1,M}(p_i)$ is the Lommel function, Erdelyi[15].

Now to obtain u, first invert the Hankel transform using the Hankel inversion formula, Tranter[16], to get

$$\bar{u}(r, s) = 2\sum_{i=1}^{\infty} \frac{(\tau' + s)\,\bar{f}(s)}{s^2 + s(\tau' + \beta + p_i^2) + \tau' p_i^2} \times \left[-J_{M-1}(p_i)\, S_{1,M}(p_i)\, \frac{J_M(p_i r)}{p_i [J_M'(p_i)]^2}\right] \quad (20)$$

the summation being over the positive roots of the Eq. (18).

Finally invert the Laplace transform using the convolution theorem, we get

$$u(r, t) = 2\sum_{i=1}^{\infty}\left[\left\{\int_0^t f(t-x)\{(\alpha_1 + \tau')\exp(\alpha_1 x) - (\alpha_2 + \tau')\exp(\alpha_2 x)\}dx\right\}\left\{\frac{(-1)J_{M-1}(p_i) S_{1,M}(p_i)}{p_i(\alpha_1 - \alpha_2)} \cdot \frac{J_M(p_i r)}{[J_M'(p_i)]^2}\right\}\right], \quad (21)$$

where

$$\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} = -\tfrac{1}{2}\left[(\tau' + \beta + p_i^2) \mp \{(\tau' + \beta + p_i^2)^2 - 4\tau' p_i^2\}^{1/2}\right] \quad (22)$$

Substitute the value of $\bar{u}$ from Eq. (20) in (15) and then invert the Laplace transform, to get the expression for the velocity of the particles

$$v(r, t) = 2\sum_{i=1}^{\infty}\left[\left\{\int_0^t f(t-x)\{\exp(\alpha_1 x) - \exp(\alpha_2 x)\}dx\right\} \times \left\{\frac{(-1)J_{M-1}(p_i) S_{1,M}(p_i)}{p_i(\alpha_1 - \alpha_2)}\right\} \cdot \frac{J_M(p_i r)}{[J_M'(p_i)]^2}\right]. \quad (23)$$

(ii) Annulus

Here we shall discuss the same problem when the flow is taking place in the annular space between two concentric circular cylinders of radii a and b ($a<b$). For the present geometry, the equations to represent the motion are (10) and (11), and the initial and boundary conditions are

Initial conditions :

$$\left.\begin{aligned} u(r, 0) &= 0 \\ v(r, 0) &= 0 \end{aligned}\right\} \quad (24)$$

Boundary conditions :

$$\left.\begin{aligned} u(1,t) &= 0 \\ u(\sigma,t) &= 0 \end{aligned}\right\} \quad (25)$$

where $\sigma = b/a$.

In this case using the Laplace transform under the initial conditions (24) and solving the resulting equations for $\bar{u}$ and $\bar{v}$, we get

$$s\bar{u} = \bar{f}(s) + \left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{M^2}{r^2}\bar{u}\right) - \frac{\beta s}{\tau' + s}\bar{u} \quad (26)$$

and

$$\bar{v} = \frac{\tau'}{\tau' + s}\bar{u}. \quad (27)$$

With the transformed boundary conditions

$$\left.\begin{aligned} \bar{u}(1,s) &= 0 \\ \bar{u}(\sigma,s) &= 0 \end{aligned}\right\} \quad (28)$$

We shall solve the eq. (26) with the help of the finite Hankel transform defined by

$$\bar{\bar{u}}(q_i, s) = \int_1^{\sigma} \bar{u}(r,s)\, r\, B_M(q_i r)\, dr, \quad (29)$$

where

$$B_M(q_i r) = J_M(q_i r) Y_M(q_i) - Y_M(q_i r) Y_m(q_i), \quad (30)$$

q_i is a positive root of the equation

$$J_M(q_i \sigma) Y_M(q_i) - Y_M(q_i \sigma) J_M(q_i) = 0 \quad (31)$$

and $J_M(q_i, r) . Y_M(q_i r)$ are the Bessel functions of first and second kind of order M and argument $q_i r$ respectively.

Applying the Hankel transform defined by Eq. (29) to (26) and on using the transformed boundary conditions (28), we get

$$s\bar{\bar{u}} = f(s)\left[\int_1^{\sigma} r\, B_M(q_i r)\, dr\right] - q_i^2\, \bar{\bar{u}} - \frac{\beta s}{\tau' + s}\bar{\bar{u}}. \quad (32)$$

Evaluating the integrals involving in the above equation with the help of a known result, Erdèlyi[15].

$$\int r^{\lambda} \zeta_M(q_i r)\, dr = \frac{r}{q_i^{\lambda}}\left[(\lambda + M - 1)\zeta_M(q_i r) S_{\lambda-1, M-1}(q_i r) - \zeta_M(q_i r) S_{\lambda, M}(q_i r)\right] \quad (33)$$

where λ is an arbitrary constant and $\zeta_M(q_i r)$ represents either of the Bessel function of first or second kind and $S_{\lambda, M}(q_i r)$ is the Lommel function, it can easily be shown that

$$\int_1^{\sigma} r\, B_M(q_i r)\, dr = \frac{2}{\pi q_i^2}\left[\frac{J_M(q_i)}{J_M(q_i \sigma)} S_{1,M}(q_i \sigma) - S_{1,M}(q_i)\right]. \quad (34)$$

Substituting Ep. (34) in (32) and then proceeding exactly on similar lines as in the first problem, we get

$$u(r,t) = \pi \sum_{i=1}^{\infty} \frac{J_M(q_i \sigma)\{J_M(q_i) S_{1,M}(q_i \sigma) - J_M(q_i \sigma)\, S_{1,M}(q_i)\}\, B_M(q_i r)}{(\beta_1 - \beta_2)\{J_M^2(q_i) - J_M^2(q_i \sigma)\}} \times \left[\int_0^t f(t-x)\{(\beta_1 + \tau')\exp(\beta_1 x) - (\beta_2 + \tau')\exp(\beta_2 x)\}\, dx\right] \quad (35)$$

and

$$v(r,t) = \pi \sum_{i=1}^{\infty} \frac{J_M(q_i \sigma)\{J_M(q_i) S_{1,M}(q_i \sigma) - J_M(q_i \sigma) S_{1,M}(q_i)\}\, B_M(q_i r)}{(\beta_1 - \beta_2)\{J_M^2(q_i) - J_M^2(q_i \sigma)\}} \times \left[\int_0^t f(t-x)\{\exp(\beta_1 x) - \exp(\beta_2 x)\}\, dx\right], \quad (36)$$

where

$$\left.\begin{matrix}\beta_1 \\ \beta_2\end{matrix}\right\} = -\tfrac{1}{2}\left[(\tau' + \beta + q_i^2) \mp \left\{(\tau' + \beta + q_i^2)^2 - 4\tau' q_i^2\right\}^{1/2}\right] \quad (37)$$

and the summation $i = 1$ to in equations (35) and (36) being over the positive roots of the equation (31).

(iii) Circular Sector

Now we consider in the present case the same problem when the flow is taking place in a cylinder whose cross-section is a sector bounded by two radii $\theta = \pm\alpha$ and the circle $R = a$ (Fig. 1). For the present geometry equation to represent the motion are (8) and (9) and the initial and boundary conditions are

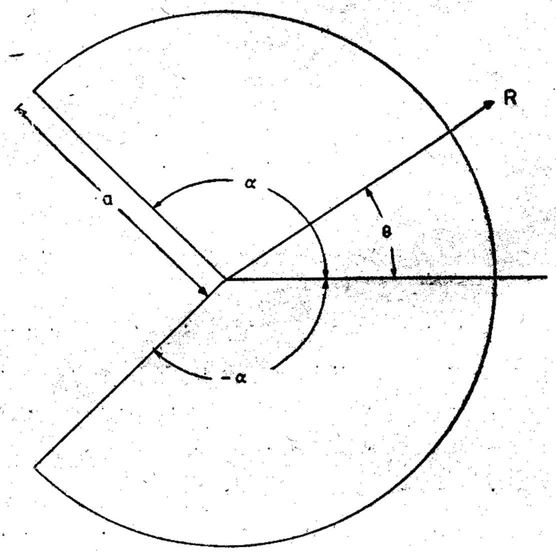

Fig. 1—Geometry of cross-section and coordinate system (Case 3).

Initial conditions :

$$\left.\begin{aligned} u(r,\theta,0) &= 0 \\ v(r,\theta,0) &= 0 \end{aligned}\right\} \quad (38)$$

Boundary conditions :

$$\left.\begin{aligned} u(1,\theta,t) &= 0 \\ u(0,\theta,t) &= 0 \end{aligned}\right\} \quad (39)$$

$$u(r, \pm\alpha, t) = 0 \quad (40)$$

In view of equation (40) it is evident that the flow is symmetrical about planes $\theta = 0$, therefore the flow in the region $0 \leqslant \theta \leqslant \alpha$ is considered and accordingly the Eq. (40) changes to

$$\left.\begin{aligned} u(r,\alpha,t) &= 0 \\ \frac{\partial u}{\partial \theta} &= 0 \text{ for } \theta = 0 \end{aligned}\right\} \quad (41)$$

To solve the problem, first apply the Laplace transform to equations (8) and (9), and solve the resulting equations for $\bar{u}$ and $\bar{v}$ and then apply the finite cosine transform defined by

$$\bar{\bar{u}}(r, m, s) = \int_0^a \bar{u}(r,\theta,s)\cos\lambda_m\theta\, d\theta, \quad (42)$$

where

$$\lambda_m = \frac{2m+1}{2\alpha}\pi \quad (43)$$

to the resulting equation for $\bar{u}$ under the transformed boundary conditions (41), to get

$$s\bar{\bar{u}} \quad \frac{(-1)^m}{\lambda_m}\bar{f}(s) + \left(\frac{\partial^2\bar{\bar{u}}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\bar{u}}}{\partial r} - \frac{M^2+\lambda^2}{r^2}\bar{\bar{mu}}\right) - \frac{\beta s}{\tau' + s}\bar{\bar{u}}. \quad (44)$$

Now proceeding exactly on similar lines as in the first problem we obtain the expressions for the velocities of fluid and particles as

$$u(r,\theta,t) = \frac{4}{\alpha}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^{m+1} J_{q-1}(\xi_i)\, S_{1,q}(\xi_i)}{\lambda_m\,\xi_i\,(\gamma_1-\gamma_2)}$$

$$\times\left[\left\{\int_0^t f(t-x)\left\{(\gamma_1+\tau')\exp(\gamma_1 x) - (\gamma_2+\tau')\exp(\gamma_2 x)\right\} dx\right\} \cdot \frac{J_q(\xi_i r)\cos\lambda_m\theta}{\left[J'\lambda_m(\xi_i)\right]^2}\right], \quad (45)$$

$$v(r,\theta,t) = \frac{4}{\alpha}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^{m+1} J_{q-1}(\xi_i)\, S_{1,q}(\xi_i)}{\lambda_m\,\xi_i(\gamma_1-\gamma_2)} \times$$

$$\left\{\left[\int_0^t f(t-x)\left\{\exp(\gamma_1 x) - \exp(\gamma_2 x)\right\} dx\right] \times \frac{J_q(\xi_i r)\cos\lambda_m\theta}{[J'_q(\xi_i)]^2}\right\}, \quad (46)$$

where

$$\left.\begin{matrix}\gamma_1 \\ \gamma_2\end{matrix}\right. = -\frac{1}{2}\left[(\tau' + \beta + \xi_i^2) \mp \{(\tau' + \beta + \xi_i^2)^2 - 4\tau'\xi_i^2\}^{1/2}\right], \quad (47)$$

$$q^2 = M^2 + \lambda_m^2 \quad (48)$$

and ξ_i is a positive root of the equation

$$J_q(\xi_i) = 0. \quad (49)$$

The summation $i=1$ to ∞ in equations (45) and (46) being over the positive roots of equation (49).

(*iv*) *Annular Sector*

In the last case, we have investigated the same problem when the flow is taking place in a cylinder whose cross-section is an annular sector bounded by two radii $\theta=\pm\alpha$ and the circles $R=a$ and $R=b$, $(a<b)$ as shown in Fig. 2. For the present problem equations (8) and (9) represent the motion and the initial and boundary conditions are

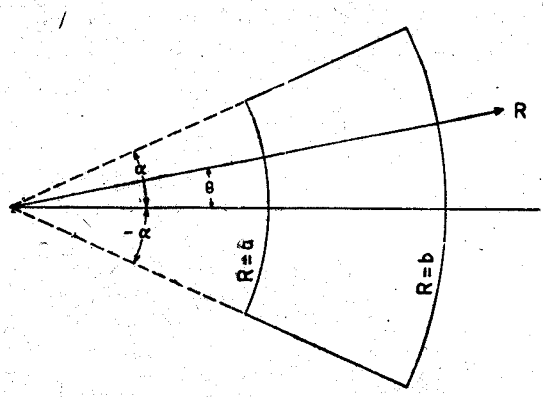

Fig. 2—Geometry of cross section (Case 4) and coordinate system.

Initial conditions :

$$\left.\begin{aligned} u(r,\theta,0) &= 0 \\ v(r,\theta,0) &= 0 \end{aligned}\right\} \tag{50}$$

Boundary conditions :

$$\left.\begin{aligned} u(1,\theta,t) &= 0 \\ u(\sigma,\theta,t) &= 0 \end{aligned}\right\} \tag{51}$$

$$u(r,\pm\alpha,t) = 0 \tag{52}$$

The solution of the problem in view of initial and boundary conditions (50) to (52) can be obtained as in previous problems (viz. with the help of Laplace transform, finite cosine transform and the finite Hankel transform for the range 1 to σ). The respective expressions for the velocity of fluid and particles are

$$u(r,\theta,t) = \frac{2\pi}{\alpha}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^m J_q(\eta_i\sigma)\{J_q(\eta_i)S_{i,q}(\eta_i\sigma) - J_q(\eta_i\sigma)S_{1,q}(\eta_i)\}}{\lambda_m(\delta_1-\delta_2)\{J_q^2(\eta_i) - J_q^2(\eta_i\sigma)\}} \times$$

$$\times\left[\int_0^t f(t-x)\{(\delta_1+\tau')\exp(\delta_1 x) - (\delta_2+\tau')\exp(\delta_2 x)\}\,dx\right]B_q(\eta_i r)\cos\lambda_m\theta, \tag{53}$$

$$v(r,\theta,t) = \frac{2\pi}{\alpha}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^m J_q(\eta_i\sigma)\{J_q(\eta_i)S_{1,q}(\eta_i,\sigma) - J_q(\eta_i\sigma)S_{1,q}(\eta_i)\}}{\lambda_m(\delta_1-\delta_2)\{J_q^2(\eta_i) - J_q^2(\eta_i\sigma)\}} \times$$

$$\times\left[\int_0^t f(t-x)\{\exp(\delta_1 x) - \exp(\delta_2 x)\}\,dx\right]B_q(\eta_i r)\cos\lambda_m\theta, \tag{54}$$

where

$$\begin{matrix}\delta_1\\ \delta_2\end{matrix} = -\frac{1}{2}\left[(\tau'+\beta+\eta_i^2) \mp \{(\tau'+\beta+\eta_i^2) - 4\tau'\eta_i^2\}^{1/2}\right], \tag{55}$$

$$B_q(\eta_i r) = J_q(\eta_i r)\,Y_q(\eta_i) - Y_q(\eta_i r)\,J_q(\eta_i) \tag{56}$$

and η_i is a positive root of the equation

$$J_q(\eta_i\sigma)\,Y_q(\eta_i) - Y_q(\eta_i\sigma)\,J_q(\eta_i) = 0. \tag{57}$$

SPECIAL CASE

The expressions for u and v obtained in the above sections are of general character and they are not restricted to any special form of the pressure gradient. As an illustrative example of the main results, we have considered in the following discussions, the flow of the conducting fluid in the coaxial cylinder when the flow is taking place under the influence of a constant pressure gradient C. Substituting $f(t)=C$ in equations (35) and (36) and on simplifying further, we get

$$u(r,t) = \pi C \sum_{i=1}^{\infty} \frac{J_M(q_i\sigma)\{J_M(q_i) S_{1,M}(q_i\sigma) - J_M(q_i\sigma) S_{1,M}(q_i)\} B_M(q_i r)}{q_i^2\{J_M^2(q_i) - J_M^2(q_i\sigma)\}} \times$$

$$\times \left[1 - \frac{(\beta_1 + q_i^2)\exp(\beta_2 t) - (\beta_2 + q^2_i)\exp(\beta_1 t)}{\beta_1 - \beta_2}\right] \tag{58}$$

and

$$v(r,t) = \pi C \sum_{i=1}^{\infty} \frac{J_M(q_i\sigma)\{J_M(q_i) S_{1,M}(q_i\sigma) - J_M(q_i\sigma) S_{1,M}(q_i)\} B_M(q_i r)}{q_i^2\{J_M^2(q_i) - J_M^2(q_i\sigma)\}} \times$$

$$\times \left[1 - \frac{\beta_1 \exp(\beta_2 t) - \beta_2 \exp(\beta_1 t)}{\beta_1 - \beta_2}\right]. \tag{59}$$

To express Eqs. (58) and (59) in further simplified form, we note that the Fourier-Bessel expansion for a function $f(r)$ over the range 1 to σ is given by, Tranter[16]

$$f(r) = \sum_{i=1}^{\infty} A_i B_M(q_i r), \tag{60}$$

where the summation being over the positive roots of equation (31) and the coefficients A_i are given by

$$A_i = \frac{\pi^2 q_i^2 J_M^2(q_i\sigma)}{2\{J_M^2(q_i) - J_M^2(q_i\sigma)\}} \left[\int_1^{\sigma} f(r)\, r\, B_M(q_i r)\, dr\right]. \tag{61}$$

Expressing r^M, r^{-M} and r^2 in terms of Fourier-Bessel expansion (60) and evaluating the integrals therein with the help of equation (33) and on simplifying further it can be shown that

$$\pi \sum_{i=1}^{\infty} \frac{J_M(q_i\sigma)\{J_M(q_i) S_{1,M}(q_i\sigma) - J_M(q_i\sigma) S_{1,M}(q_i)\} B_M(q_i r)}{q_i^2\{J_M^2(q_i) - J_M^2(q_i\sigma)\}}$$

$$= \frac{1}{4-M^2}\left[\frac{\sigma^{-M} - \sigma^2}{\sigma^{-M} - \sigma^M} r^M + \frac{\sigma^M - \sigma^2}{\sigma^M - \sigma^{-M}} r^{-M} - r^2\right] \tag{62}$$

In view of (62), equations (58) and (59) become

$$u(r,t) = \frac{C}{4-M^2}\left[\frac{\sigma^{-M}-\sigma^2}{\sigma^{-M}-\sigma^M}\; r^M + \frac{\sigma^M-\sigma^2}{\sigma^M-\sigma^{-M}}\, r^{-M} - r^2\right] -$$

$$- \pi C \sum_{i=0}^{\infty} \frac{J_M(q_i\sigma)\,[\,J_M(q_i)\,S_{1,M}(q_i\sigma) - J_M(q_i\sigma)\,S_{1,M}(q_i)\,]\,B_M(q_i r)}{q_i^2\,[\,J_M^2(q_i) - J_M^2(q_i\sigma)\,]} \times$$

$$\times \left[\frac{(\beta_1+q_i^2)\exp(\beta_2 t) - (\beta_2+q_i^2)\exp(\beta_1 t)}{\beta_1-\beta_2}\right] \tag{63}$$

and

$$v(r,t) = \frac{C}{4-M^2}\left[\frac{\sigma^{-M}-\sigma^2}{\sigma^{-M}-\sigma^M}\; r^M + \frac{\sigma^M-\sigma^2}{\sigma^M-\sigma^{-M}}\, r^{-M} - r^2\right] -$$

$$- \pi C \sum_{i=1}^{\infty} \frac{J_M(q_i\sigma)\,[J_M(q_i)\,S_{1,M}(q_i\sigma) - J_M(q_i\sigma)\,S_{1,M}(q_i)\,]\,B_M(q_i r)}{q_i^2\,[\,J_M^2(q_i) - J_M^2(q_i\sigma)\,]} \times$$

$$\times \left[\frac{\beta_1\exp(\beta_2 t) - \beta_2\exp(\beta_1 t)}{\beta_1-\beta_2}\right] \tag{64}$$

Though the equations (63) and (64) become indeterminate when $M=0$ or 2, yet they tend to finite values as $M\to 0$ or 2. Thus letting $M\to 0$ and 2 and using the known result, Erdélì[15].

$$S_{1,0}(\lambda) = 1,\; S_{1,2}(\lambda) = (\lambda^2 - 4), \tag{65}$$

these equations become

$$u(r,t) = \frac{C}{4}\left[\,(1-r^2) - (1-\sigma^2)\,\frac{\log r}{\log\sigma}\,\right] - \pi C \sum_{i=1}^{\infty} \frac{J_0(q_i\sigma)\,B_0(q_i r)}{q_i^2\,[J_0(q_i) + J_0(q_i\sigma)\,]} \times$$

$$\times\; \frac{(\beta_1+q_i^2)\exp(\beta_2 t) - (\beta_2+q_i^2)\exp(\beta_1 t)}{\beta_1-\beta_2}\,, \tag{66}$$

$$v(r,t) = \frac{C}{4}\left[\,(1-r^2) - (1-\sigma^2)\,\frac{\log r}{\log\sigma}\,\right] - \pi C \sum_{i=1}^{\infty} \frac{J_0(q_i\sigma)\,B_0(q_i r)}{q_i^2\,[J_0(q_i) + J_0(q_i\sigma)\,]} \times$$

$$\times \left[\frac{\beta_1\exp(\beta_2 t) - \beta_2\exp(\beta_1 t)}{\beta_1-\beta_2}\right]. \tag{67}$$

$$u(r,t) = \frac{C}{4}\left[\frac{\sigma^4(r^4-1)}{r^2(\sigma^4-1)}\log\sigma - r^2\log r\right] -$$

$$-\pi C\sum_{i=1}^{\infty}\frac{J_2(q_i\sigma)[(q_i^2\sigma^2-4)J_2(q_i)-\sigma(q_i^2-4)J_2(q_i\sigma)]B_2(q_i r)}{\sigma q_i^3[J_2^2(q_i)-J_2^2(q_i\sigma)]}\times$$

$$\times\left[\frac{(\beta_1+q_i^2)\exp(\beta_2 t)-(\beta_2+q_i^2)\exp(\beta_1 t)}{\beta_1-\beta_2}\right] \tag{68}$$

and

$$v(r,t) = \frac{C}{4}\left[\frac{\sigma^4(r^4-1)}{r^2(\sigma^4-1)}\log\sigma - r^2\log r\right] -$$

$$-\pi C\sum_{i=1}^{\infty}\frac{J_2(q_i\sigma)[(q_i^2\sigma^2-4)J_2(q_i)-\sigma(q_i^2-4)J_2(q_i^2\sigma)]B_2(q_i r)}{\sigma q_i^3 J_2^2(q_i)-J_2^2(q_i)}\times$$

$$\times\left[\frac{\beta_1\exp(\beta_2 t)-\beta_2\exp(\beta_1 t)}{\beta_1-\beta_2}\right] \tag{69}$$

respectively.

Equations (66) and (67) represent the velocities of the fluid and particle in the absence of magnetic field.

Now making use of Bessel's inequality, which states that for an orthonormal set $B_M(q_i r)$, whether closed or not, we have

$$\sum_{i=1}^{\infty}A_i^2 \leqslant \int_1^{\sigma}\left[\phi(q_i r)\right]^2 dr, \tag{70}$$

where A_i^2 are the coefficients in the generalised Fourier's expansion of $\phi(q_i r)$ in terms of $B_M(q_i r)$ and $(1, \sigma)$ is the interval of orthonormality. Evidently the transient part T in Eq. (63) satisfies

$$\left|T\right| \leqslant \pi C\sum_{i=1}^{\infty}\frac{J_M(q_i\sigma)[J_M(q_i)S_{1,M}(q_i\sigma)-J_M(q_i\sigma)S_{1,M}(q_i)]B_M(q_i r)}{q_i^2[J_M^2(q_i)-J_M^2(q_i\sigma)]}\times$$

$$\times\left[\frac{-(\beta_1'+q_1^2)\exp(\beta_2' t)+(\beta_2'+q_1^2)\exp(\beta_1' t)}{\beta_1'-\beta_2'}\right], \tag{71}$$

where β_1' and β_2' are the values of β_1 & β_2 at the smallest root q_1 of equation (31).

Hence from equations (63) and (58), we have

$$u(r,t) \cong \frac{C}{4-M^2}\left[\frac{\sigma^{-M}-\sigma^2}{\sigma^{-M}-\sigma^M}r^M\ \frac{\sigma^M-\sigma^2}{\sigma^M-\sigma^{-M}}r^{-M}-r^2\right]\times$$

$$\times\left[1-\frac{(\beta_1'+q_1^2)\exp(\beta_2' t)-(\beta_2'+q_1^2)\exp(\beta_1' t)}{\beta_1'-\beta_2'}\right]. \tag{72}$$

Following the same procedure the expression for the velocity of the particles is

$$v(r,t) \cong \mid \frac{C}{4-M^2}\left[\frac{\sigma^{-M}-\sigma^2}{\sigma^{-M}-\sigma^{M}}\, r^M + \frac{\sigma^M-\sigma^2}{\sigma^M-\sigma^{-M}}\, r^{-M} - r^2\right]\times$$

$$\times\left[1-\frac{\beta_1'\exp(\beta_2' t)-\beta_2'\exp(\beta_1' t)}{\beta_1'-\beta_2'}\right]. \qquad (73)$$

Mean velocity

The dimensionless mean velocities are given by

$$u^* = \frac{\int_0^{2\pi}\int_1^{\sigma} u.r\,d\theta\,dr}{\int_0^{2\pi}\int_1^{\sigma} r\,d\theta\,dr} \qquad (74)$$

$$v^* = \frac{\int_0^{2\pi}\int_1^{\sigma} v.r\,d\theta\,dr}{\int_0^{2\pi}\int_1^{\sigma} r\,d\theta\,dr} \qquad (75)$$

Thus Eqs. (63) and (64) or (72) and (73) may be integrated to find the mean velocities for a constant pressure gradient. The numerical results of the mean velocities are shown in Fig. 7.

DISCUSSION

It is interesting to observe from Eqs. (45), (46) and (53), (54) that if we let the sectorial angle be 2π the resulting solutions do not correspond to the flow through the circular cylinder and through the annulus. It is because in these cases we shall have a semi-diametral wall in the cylinder extending along the length of the cylinder and joining the axis with one of the generators of the circular boundary.

In the Figs. 3 to 6, we have drawn the velocity profiles of the fluid and particles when flow takes place under the influence of a constant pressure gradient C through the annulus. Fig. 3 shows the velocity profiles of the fluid and particles in the absence of magnetic field, while Figs. 4 to 6 represent the velocity profiles for values of M as indicated in figures. From these figures it is observed that the fluid moves faster than the particles and as time t increases the velocities approach

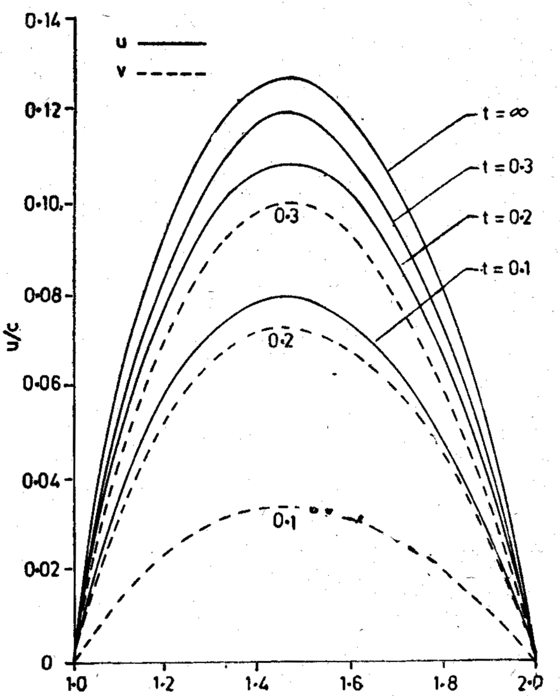

Fig. 3—Velocity profile of fluid and particles for values of t indicated when $f = 0.1$, $\tau = 0.1$ and $M = 0$

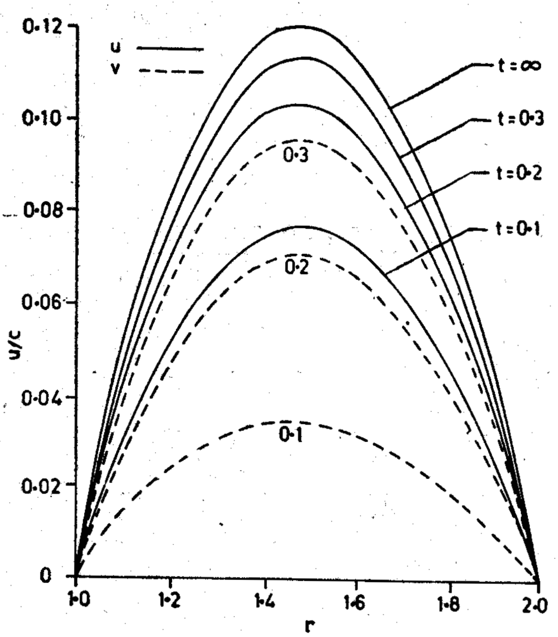

Fig. 4—Velocity profile of fluid and particles for values of t indicated when $f = 0.1$, $\tau = 0.1$ and $M = 1$.

the steady states. With increase in the magnetic field the velocities of the fluid and particles decreases. For larger values of M the velocity profiles become more flattened.

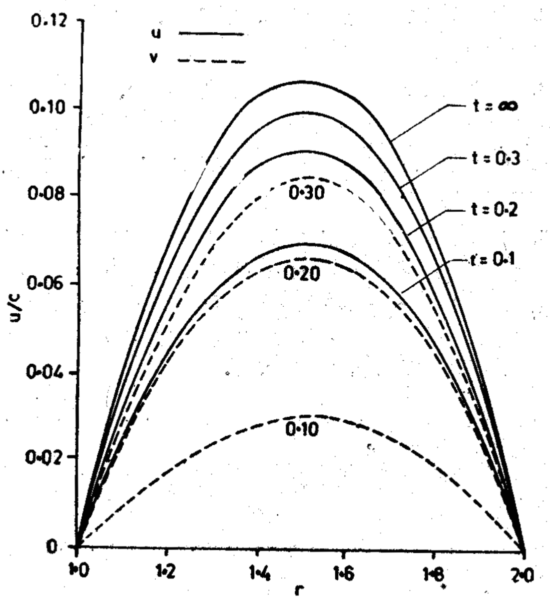

Fig. 5—Velocity profile of fluid and particles for values of t indicated when $f = 0.1$, $\tau = 0.1$ and $M = 2$.

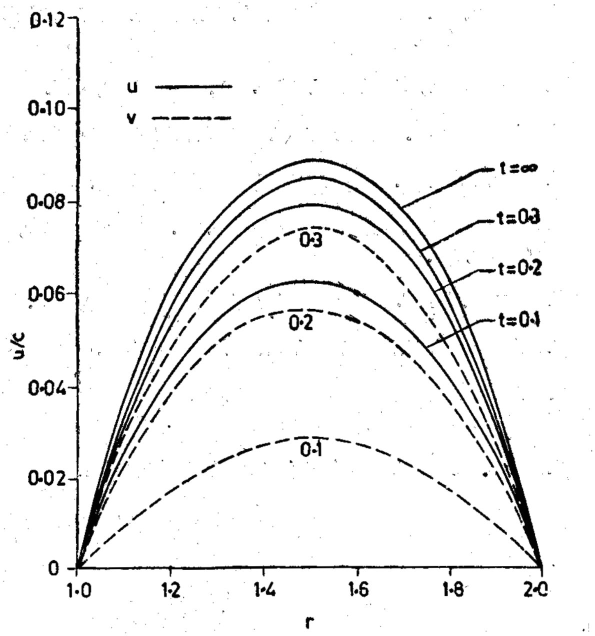

Fig. 6—Velocity profile of fluid and particles for values of t indicated when $f = 0.1$, $\tau = 0.1$ and $M=3$.

Fig. 7 shows some typical examples of the results obtained. It is observed from this figure that the fluid-particles velocity difference is small when τ is small and large when τ is large. At low particle concentration, the fluid moves the particles with it at all times when τ is small. But with large τ, the fluid flow is unaffected, but it results in large fluid-particle velocity difference. Large particle concentrations result in lower particle and fluid acceleration. For larger τ, the fluid starts accelerating sooner but the particles start later, and the time to reach steady flow is also increased. Finally the fluid and the particles accelerate to the same final steady state.

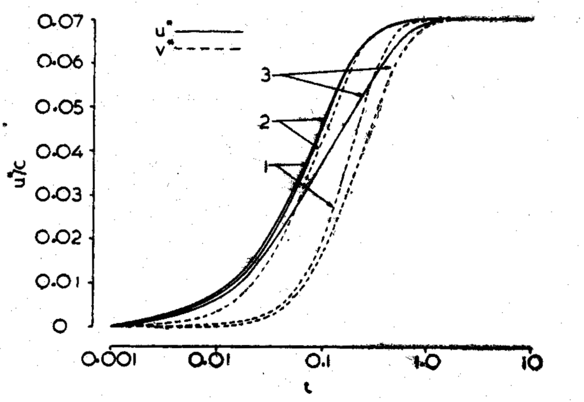

Fig. 7—Mean velocity of the fluid and particles for (1) $f = 0.10$, $\tau = 0.10$; (2) $f = 0.10$, $\tau = 0.01$ and (3) $f = 1.0$, $\tau = 0.10$ and $M = 2$.

In the absence of magnetic field i.e., when $M \to 0$ the results (21), (23) and (45), (46) corresponds to the results, recently obtained by Gupta & Gupta.[13]

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REFERENCES

1. Gold, R. R., *J. Fluid Mech.*, **13** (1962), 505.
2. Singh, D. & Rizvi Syed, A. T., *Phys. Fluids*, **7** (1964), 760.
3. Singh, D., *Appl. Sci. Res. B.*, **10** (1963), 413.
4. Saffman, P. G., *J. Appl. Mech.*, **13** (1962), 120.
5. Liu, J. T. C., *Phys. Fluid*, **9** (1966), 1716.
6. Michael, D. H. & Miller, D. A., *Mathematika*, **13** (1966), 97.
7. Liu, J. T. C., *Astronautica Acta.*, **13** (1967), 369.
8. Vimla, C. S., *Def. Sci. J.*, **22** (1972), 231.
9. Verma, P. D. & Mathur, A. K., *Ind. J. Pure Appl. Math.*, **4** (1973), 133.
10. Rao, P. S. S., *Def. Sci. J.*, **19** (1969), 135.
11. Singh, J. P. & Pathak, R. S., *Ind. J. Pure Appl. Math.*, **8** (1977), 314.
12. Gupta, R. K. & Gupta, S. C., *J. Appl. Math. & Phy.*, **27** (1976), 119.
13. Gupta, R. K. & Gupta S. C., *Proc. Indian Acad. Sci.*, **86 A** (1977), 591.
14. Yang, H. T. & Healy, J. V., *Appl. Sci. Phy.*, **27** (1978), 387.
15. Erdelyi, A., *et. al.* 'Higher Transcendental Function', (Mc Graw Hill Book Co., New York), Vol. II, 1953, p. 90.
16. Tranter, C. J., 'Bessel Functions with Some Physical Applications', (The English University Press Ltd., London), Vol. I, 1968, p. 65.
17. Prakash, S., *J. Appl. Mech. Trans. ASME*, Ser. E, **36** (1969), 309.