BEHAVIOUR OF SONIC WAVES IN DISSOCIATING GASES IN THE PRESENCE OF A MAGNETIC FIELD

B. G. VERMA & R. C. SRIVASTAVA

Department of Mathematics

University of Gorakhpur, Gorakhpur-273001

(*Received* 14 *September* 1977)

This paper studies sonic waves in dissociating gases under the influence of a magnetic field. The conductivity of the medium has been taken to be finite since the temperature range considered is 1000°K to 7000°K. Since bodies travelling with hypersonic velocities meet with the phenomena of dissociation, this paper has a direct relevance to defence problems.

Thomas[1], Nariboli & Secrest[2] and many other authors have made intensive investigations regarding the propagation of sonic waves in ordinary as well as conducting gases. Recently, Srinivasan & Ram[3] studied the growth and decay behaviour of sonic waves in radiating gases. When projectiles travel with speed of order 2 km/sec, regions adjacent to the body of the projectile meet with the phenomena of dissociation. To simplify the analysis of dissociation, Lighthill[4] introduced the concept of an ideal dissociating gas. In the present work, we study the behaviour of sonic waves in a dissociating gas of Lighthill's model under a magnetic field. We consider the temperature range 1000°K to 7000°K in which dissociation is important but electronic excitation energy and ionization energy are negligible[5]. Accordingly, the conductivity of the medium cannot be taken to be infinite. In order to avoid the complexities that arise from the detailed composition of air, we take only diatomic gases. As a result of the discussion that follows we obtain that small disturbances produced in the ideal dissociating gas under a magnetic field travel with a velocity which is a combination of the effective velocity of sound, Alfven velocity as well as a term having dimensions of velocity and occurring on account of the finite conductivity of the medium. We have also shown that Alfven velocity depends on the degree of dissociation and sonic waves rapidly terminate into a shock wave.

FORMULATION OF THE PROBLEM

Consider the diatomic gas mixture consisting of the same kind of molecules A_2, composed of identical atoms A. Suppose that, at temperature T and density ρ, a fraction α of the original number of molecules dissociates into atoms by the reaction

$$A_2 \underset{K_r}{\overset{K_f}{\rightleftharpoons}} 2A$$

where K_f and K_r are reaction rate constants.

$$p = (1 + \alpha)\, \rho\, RT, \qquad h = (4 + \alpha)\, RT + \alpha D \tag{1}$$

The equation of continuity due to Lighthill[4] can be written as

$$\frac{\partial \alpha}{\partial t} + U_{\alpha,\, i} = \frac{4 \rho\, D' K_r\, (1 + \alpha)}{R'\, T_d^{2}} \left\{ \rho_d\ (1 - \alpha)\ \exp \left(- \frac{T_d}{T} \right) - \rho\, \alpha^2 \right\} \tag{2}$$

where ρ_d, T_d, D and R are respectively characteristic density, characteristic temperature of dissociation, dissociation energy per unit mass and gas constant for A_2. Although ρ_d is a function of T its variation in the temperature range under consideration is very small. Therefore, we take ρ_d as constant as this assumption hardly affects the result.

We consider the flow of an ideal dissociating gas of finite electrical conductivity in the presence of a magnetic field. Using cartesian tensor notation in M.K.S. units, the relevant equations are[6]

$$\frac{\partial \rho}{\partial t} + U_i \rho_{,i} + \rho U_{i,i} = 0 \tag{3}$$

$$\rho \frac{\partial U_i}{\partial t} + \rho U_j U_{i,j} + p_{,i} + H_j H_{j,i} - H_j H_{i,j} = 0 \tag{4}$$

$$\frac{\partial H_i}{\partial t} + U_j H_{i,j} - H_j U_{i,j} + H_i U_{j,j} - \frac{1}{\sigma} H_{i,jj} = 0 \tag{5}$$

$$\rho \frac{\partial h}{\partial t} + \rho U_i h_{,i} - \frac{\partial p}{\partial t} - U_i p_{,i} = \frac{J^2}{\sigma} \tag{6}$$

where

$$J_i = e_{ijk} H_{k,j}$$

and e_{ijk} is the permutation tensor.

The quantities U_i, p, ρ, h and H_i denote respectively the gas velocity components, pressure, density, enthalpy of the gas mixture and magnetic field components.

In view of the Eqns. (1) and (2), Eqn. (6) can be written as

$$\frac{\partial p}{dt} - \rho U_i \frac{\partial U_i}{\partial t} - \rho U_i U_j U_{i,j} + \gamma_e p U_{i,i} + F(p, \rho, \alpha) = \frac{J^2}{\sigma} \tag{7}$$

where

$$F(p, \rho, \alpha) = \frac{4 \rho D^2 K_r}{3R^2 T_d^2} \left\{ 3p - \rho D (1+\alpha)^2 \right\} \left\{ \rho_d (1-\alpha) \exp\left(-\frac{T_d}{T}\right) - \rho \alpha^2 \right\}$$

and

$$\gamma_e = \frac{4+\alpha}{3}$$

is the effective exponent of heat for the gas mixture.

VELOCITY OF SONIC WAVE

Suppose that a surface $\Sigma(t)$ of a sonic wave, across which the flow parameters are continuous but their derivatives of all order are discontinuous, moves with a velocity G. We further assume that across this surface, the magnetic field and its second and higher order derivatives are continuous while its first order derivative is discontinuous. Then, taking jumps of Eqns. (2) to (6) and making use of the geometrical and kinematical compatibility conditions of Thomas[7], we get

$$(U_n - G)[\alpha_{,i}] n_i = 0 \tag{8}$$

$$(U_n - G)\zeta + \rho \lambda_i n_i = 0 \tag{9}$$

$$(U_n - G)\lambda_i + \mu n_i + H_j \xi_j n_i - H_n \xi_i = 0 \tag{10}$$

$$(U_n - G)\xi_i - H_j n_j \lambda_i + H_i \lambda_j n_j = 0 \tag{11}$$

$$\rho (U_n - G)\lambda_i U_i + G\mu - \gamma_e p \lambda_i n_i - \frac{E}{\sigma} \xi_i^2 = 0 \tag{12}$$

where [ ] indicates the discontinuity in the quantity enclosed and

$$\lambda_i = [U_{i,j}] n_j, \quad \mu = [p_{,i}] n_i, \quad \zeta = [\rho_{,i}] n_i,$$

$$\xi_i = [H_{i,j}] n_j, \quad U_n = u_i n_i$$

n_i being the components of the unit normal to the surface $\Sigma(t)$. To determine the velocity of sonic wave, we use Eqns. (10), (11) and (12) and the fact that $\lambda \neq 0$. The relative velocity $(G - U_n)$ is then given by

$$\left\{ (G - U_n)^2 - \frac{G}{G - U_n} \left(\frac{H_i^2 - H_n^2}{\rho} \right) - \frac{\gamma_e p}{\rho} \right\} - \frac{E \xi_i^2}{\rho \sigma \lambda} = 0 \tag{13}$$

From (13) it follows that when the medium ahead of Σ (t) is at rest, the wave moves with a velocity, G given by

$$G = \frac{\gamma_e p}{\rho} + \frac{H_i^2 - H_n^2}{\rho} + \left(\frac{E \xi_i^2}{\rho \sigma \lambda} \right) \tag{14}$$

We therefore conclude that small disturbances in a dissociating gas in the presence of a magnetic field travel with a velocity which is the combination of effective velocity of sound, Alfven velocity, and a velocity vector due to the finite conductivity of the medium.

Taking the magnetic pressure number $\beta = \frac{H^2{}_i}{2p}$ as constant throughout the medium, we obtain, after a slight manipulation

$$\beta = \frac{H_i^2}{2p} = \frac{[H_i H_{i,j}]}{[p,_j]} > 0$$

Consequently, we get the relations

$$H_i \xi_i = \beta \mu \tag{15}$$

$$A^2 = \frac{2 S^2 \beta}{\gamma_e} \tag{16}$$

where A is the Alfven velocity given by $A^2 = \frac{H^2{}_i}{\rho}$ and S is the sound velocity defined by $S^2 = \frac{\gamma_e p}{\rho}$.

The relation (16) implies that Alfven velocity is dependent on the degree of dissociation and increases (decreases) with its decreasing (increasing) degree. Taking the medium ahead of the surface Σ (t) at rest we get, as a consequence of the Eqns. (9), (10), (13) and (15),

$$\lambda = \frac{(1 + \beta) \mu}{G \rho} = \frac{G}{\rho} \zeta = \left(\frac{E}{\rho \sigma a^2} \right) \xi_i^2 \tag{17}$$

where we have taken

$$a^2 = G^2 - \frac{\gamma_e p}{\rho} - \frac{H_i^2 - H_n^2}{\rho}$$

BEHAVIOUR OF SONIC WAVES

Differentiating (1) with respect to x_j and making use of compatibility condition, we obtain

$$\mu = (1 + \alpha) RT \zeta + (1 + \alpha) R \rho [T,_j] n_j \tag{18}$$

In order to derive the expressions governing the growth and decay behaviour of sonic waves, we differentiate Eqns. (3) to (5) and (7) with respect to x_j. Using compatibility conditions of Thomas[7], the Eqns. (18) and the fact that $U_i=0$ on Σ (t), we get

$$\frac{\delta \zeta}{\delta t} = 2 \zeta \lambda + G \bar{\zeta} + 2 \rho \lambda \Omega - \rho \bar{\lambda}_i n_i \tag{19}$$

$$\rho \frac{\delta \lambda}{\delta t} = \rho G \lambda_i n_i - \mu \tag{20}$$

$$\frac{\delta \xi_i}{\delta t} = 2 \xi_i \lambda - (\overline{H_i - H_j}) (\lambda_i n_i - 2 \Omega \lambda) \tag{21}$$

$$\frac{\delta \mu}{\delta t} = 2 \gamma_e p \Omega \lambda + G \bar{\mu} - \gamma_e p \bar{\lambda}_i n_i + \rho G \lambda^2 + \gamma_e \mu \lambda - (A \zeta - B \mu) \tag{22}$$

where Ω is the mean curvature of $\Sigma(t)$ and

$$A = Z\{\rho D(1+\alpha)^2(2X - \alpha^2\rho) - 3p(X - \alpha^2) - Y\}$$

$$B = \rho Z(Y - 3X)$$

$$X = \rho_d(1-\alpha)\exp\left(-\frac{T_d}{T}\right) - \rho\alpha^2$$

$$Y = \left\{\rho D(1-\alpha)^2 - 3p\right\}\left\{\rho_d(1-\alpha)\exp\left(-\frac{T_d}{T}\right)\frac{T_d}{T}\right\}$$

$$Z = \frac{4D^2K_r}{3R^2T_d^2}$$

$$\bar{\lambda}_i = \left[U_{i,jk}\right]n_j n_k$$

$$\bar{\mu} = \left[p,jk\right]n_j n_k$$

$$\bar{\zeta} = \left[\rho,jk\right]n_j n_k$$

Taking the time-derivative of Eqn. (17) and combining with (19) to (22) we get, the following set of Equations

$$\frac{\delta\zeta}{\delta t} = \frac{G}{\rho}\bar{\gamma}_e\zeta^2 + P_1\zeta + \frac{1+\beta}{G^2}Q_1 \tag{23}$$

$$\frac{\delta\lambda}{\delta t} = \bar{\gamma}_e\lambda^2 + P_1\lambda + \frac{1+\beta}{\rho G}Q_1 \tag{24}$$

$$\frac{\delta\xi_i^2}{\delta t} = \bar{\gamma}_e\frac{E}{\rho\sigma a^2}\xi_i^4 + P_1\xi_i^2 + \frac{(1+\beta)\sigma a^2}{EG}Q_1 \tag{25}$$

$$\frac{\delta\mu}{\delta t} = \frac{(1+\beta)\bar{\gamma}_e}{\rho G}\mu^2 + P_1\mu + Q_1 \tag{26}$$

where

$$\bar{\gamma}_e = 1 + \beta + \gamma_e$$

$$P_1 = \frac{2(1+\beta)}{G}\frac{\bar{\gamma}_e p}{\rho}\Omega - \left(\frac{1+\beta}{G^2}A - B\right)$$

$$Q_1 = (G\bar{\mu} - \gamma_e p\,\bar{\lambda}_i n_i$$

The Eqns. (23) to (26) govern the growth and decay behaviour of sonic waves propagating in an ideal dissociating gas in the presence of magnetic field.

We now consider a sonic surface $\Sigma(t_0)$ at time t_0. If s represents the distance measured from $\Sigma(t_0)$ in the direction of normal to the surface $\Sigma(t)$, then $s = G(t - t_0)$ and scalars λ, μ, ζ and ξ_i may be regarded as functions of s. Therefore, we can write

$$\left.\begin{aligned} \frac{\delta\lambda}{\delta t} &= G\frac{d\lambda}{ds}, \quad \frac{\delta\zeta}{\delta t} = G\frac{d\zeta}{ds} \\ \frac{\delta\xi_i}{\delta t} &= G\frac{d\xi_i}{ds}, \quad \frac{\delta\mu}{\delta t} = G\frac{d\mu}{ds} \end{aligned}\right\} \tag{27}$$

In view of the Eqn. (27), the Eqns. (23) to (26) can be written in the following form

$$\frac{d\zeta}{ds} = \frac{\bar{\gamma}_e}{\rho}\zeta^2 + P\zeta + \frac{1+\beta}{G^2}Q \tag{28}$$

$$\frac{d\lambda}{ds} = \frac{\bar{\gamma}_e}{G}\lambda^2 + P\lambda + \frac{1+\beta}{\rho G}Q \tag{29}$$

$$\frac{d\xi_i^2}{ds} = \frac{\bar{\gamma}_e}{G}\frac{E}{\zeta\sigma a^2}\xi_i^4 + P\xi_i^2 + \frac{(1+\beta)\sigma a^2}{EG}Q \tag{30}$$

$$\frac{d\mu}{ds} = \frac{(1+\beta)\bar{\gamma}_e}{\rho G^2}\bar{\mu}^2 + P\mu + Q \tag{31}$$

where

$$P = \frac{P_1}{G}\ ;\ Q = \frac{Q_1}{G}$$

Since the Eqns. (28) to (31) are of the same form, only one of them is sufficient to predict the nature of the sonic waves.

Then to discuss the nature of the sonic waves, we integrate (28). To avoid complications in analysis, we assume here that the sonic surface is moving with a velocity large enough in comparison to the Alfven velocity. This permits us to neglect the last term in (28) which now takes the form

$$\frac{d\zeta}{ds} - \frac{\bar{\gamma}_e}{\rho}\zeta^2 - P\zeta = 0 \tag{32}$$

On integration, this yields

$$\frac{1}{\zeta} = \frac{1}{\zeta_0}\exp(cs)\,\triangle - \frac{\bar{\gamma}_e}{\rho}\exp(cs)\,\triangle\int_0^s \frac{\exp(-cs)}{\triangle}ds \tag{33}$$

where

$$c = \left(\frac{1+\beta}{G^3}\cdot A - \frac{B}{G}\right),\ \triangle = \left\{1 - 2\,\Omega_0\,s + K_0 s^2\right\}^{\frac{1+\beta}{G^2}\ \frac{\gamma ep}{\rho}}$$

Ω being the mean curvature of the wave front and according to Lane[8], we get

$$\Omega = \frac{\Omega_0 - K_0 s}{1 - 2\,\Omega_0 s + K_0 s^2}$$

To have a clear physical situation we consider the particular case of a plane wave. In this case the mean curvature is zero so that $\Omega = \Omega_0 = 0$ and $K_0 = 0$. Then the Eqn. (33) reduces to the form

$$\Psi = \frac{1}{\exp(\eta) - A_0\{\exp(\eta) - 1\}} \tag{34}$$

where

$$\Psi = \frac{\zeta}{\zeta_0},\ \eta = cs,\ A_0 = \frac{\bar{\gamma}_e}{\rho c}\zeta_0$$

The parameter Ψ is the measure of the strength of the shock discontinuity during the propagation, η is a variable parameter and A_0 depends on ζ_0 and the magnetic pressure number. The Eqn. (34) suggests that for a compressive wave of order one, ζ grows continuously as

$$s \to \frac{1}{c} \log \frac{\bar{\gamma}_e |\zeta_0|}{\bar{\gamma}_e |\zeta_0| - \rho_c} \tag{35}$$

provided

$$|\zeta_0| > \frac{\bar{\rho}_c}{\bar{\gamma}_e} \tag{36}$$

for a real value of s. In such a case, continuity across $\Sigma(t)$ breaks down and sonic waves terminate into a shock wave in time t_c given by

$$t_c = t_0 + \frac{1}{cG} \log \left\{ \frac{\bar{\gamma}_e |\zeta_0|}{\bar{\gamma}_e |\zeta_0| - \rho_c} \right\} \tag{37}$$

From this relation we conclude that in the presence of a magnetic field, sonic waves rapidly terminate into a shock wave. It may be remarked that inequality (36) holds for the non-magnetic case as well.

CONCLUSION

The propagation of sonic waves in an ideal dissociating gas of Lighthill's model has been discussed under a magnetic field. It has been shown that the magnetic field changes the velocity of small disturbances that now travel with a velocity which is a combination of the effective veloeity of sound, Alfven velocity and a velocity due to finite conductivity of the medium. It has been also shown that Alfven velocity depends on the degree of dissociation and sonic waves rapidly terminate into a shock wave.

REFERENCES

1. THOMAS, T. Y., *J. Math. Mech.*, **6** (1957), 455.
2. NARIBOLI, G. A. & SECREST, B. G., 'Weak Discontinuities in Magnetogasdynamics in the Presence of a Dissipative Mechanism' (Tensor, N. S.), 1967, p. 22.
3. SRINIVASAN & RAM, R., *ZAMP.*, **26** (1975), 307.
4. LIGHTHILL, M. J., *J. Fluid Mech.*, **2** (1957), 1.
5. CLARKE, J. F., *J. Fluid Mech.*, **7** (1960), 577.
6. PAI, S. I., 'Magnetogasdynamics and Plasma Dynamics' (Prentice-Hall, Inc., N. J.), 1962, p. 109.
7. THOMAS, T. Y., *J. Math. Mech.*, **6** (1957), 311.
8. LANE, L. P., 'Metric Differential Geometry of Curves and Surfaces', 1940, p. 209.