EXTENSION OF RESISTANCE LAW BY APPROXIMATED FUNCTIONS FOR TRAJECTORY

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An attempt has been made to extend the approximated function beyond 914 m/sec. with the help of suitable functions to enable the trajectory computation of the projectile without using the retardation coefficients from 1940 Resistance Law Tables. The results obtained with these functions have been compared with those obtained from Primary and Secondary, Ballistic Functions and using associated values defined in Siacci's method.

The solution of the equations of motion of the standard projectile with the simplified assumptions using primary/secondary functions involving the retardation coefficient as defined in Siacci's method is quite cumbersome. It has been found that the computation of trajectory of projectiles becomes much easier with the help of approximated functions based on 1940 Resistance Law Tables and the knowledge of numerical methods for using electronic digital computers to achieve results in an efficient way. This enables to determine the trajectory elements at any instant of flight. Due to certain limitations of using Ballistic functions and the associated values available, the Siacci's method[1,3] gives good approximation for elements at the vertex and at the point of fall.

The retardation coefficient $P\,(V)$ was prepared on the basis of extensive trials conducted in UK during 1940. Ghosh and Srinivasan[2] tried to approximate these tables by the combination of three suitable functions in the form of $G\,(W)$ valid in the three different range of velocities. They considered these tables upto the velocity of the order of 914 m/sec. approximated by the functions as follows :

$$\text{For } 1000 \leqslant W \leqslant 1210,\; G(W) = a_0 + \sum_{n=1}^{4} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } x = \left(\frac{W - 10^3}{200} \right) \pi \tag{1}$$

$$\text{For } 1210 < W \leqslant 1270,\quad G(W) = d_0 + d_1\, W \tag{2}$$

$$\text{For } 1270 < W \leqslant 3000,\quad G(W) = A + BW + CW^2 \tag{3}$$

The constants are determined on fitting the curve by the method of least squares.

$$\left.\begin{array}{ll} a_0 = +\,0.769737 & b_1 = -\,0.088167 \\ a_1 = -\,0.274767 & b_2 = -\,0.012182 \\ a_2 = -\,0.090770 & b_3 = +\,0.029389 \\ a_3 = +\,0.021537 & b_4 = +\,0.006091 \\ a_4 = +\,0.018603 & \end{array}\right\} \tag{4}$$

The straight line approximation constants are

$$\left.\begin{array}{l} d_0 \doteq \;\; 1.17736 \\ d_1 = -\,0.186 \times 10^{-3} \end{array}\right\} \tag{5}$$

The constants for the second degree polynomial curve are

$$\left.\begin{aligned} A &= 1.3916324 \\ B &= -0.41160982 \times 10^{-3} \\ C &= 0.04502329 \times 10^{-6} \end{aligned}\right\} \tag{6}$$

The error with these approximations was everywhere less than 0.2 per cent from the correct value.

There is a necessity to work out the trajectories of the projectile with velocities which are more than 914m/sec. In the present paper, the 1940 Resistance functions have been approximated with a view to extend these up to 1829 m/sec. The effect of these approximated functions on the trajectories have been studied and the results have been compared with those obtained from Ballistic functions and using associated values defined in Siacci's method.

AIR RESISTANCE LAW

The air resistance of the projectile for all practical purposes in Ballistics is written in the form[3]

$$R = \frac{\rho}{\rho_0} K\sigma\, d^2 \left(\frac{v}{100}\right)^2 P\left(\frac{v}{a}\right) \tag{7}$$

where $K\sigma$ is the coefficient of shape and steadiness, ρ_0 is the standard density, d is the calibre, v is the velocity of the projectile, ρ is the density at an altitude y, and $P\left(\frac{v}{a}\right)$, a function of Mach number which is proportional to the drag coefficient. Hence the retardation reduces to

$$r = \frac{1}{C_0 f(y)} \left(\frac{v}{100}\right)^2 P\left(\frac{v}{a}\right) \tag{8}$$

where

C_0 is the Standard Ballistic Coefficient defined as

$$C_0 = \frac{m}{K\sigma\, d^2} \tag{9}$$

m is the mass of projectile and $f(y)$ is an altitude function for the variation of density with height. It is approximated by simple function of altitude as

$$f(y) = \frac{1}{H(y)} = \frac{\rho_0}{\rho} = e^{\lambda \cdot y} \tag{10}$$

for y up to 10668 m, $\lambda = 3.2466449811 \times 10^{-5}$.

The retardation coefficient P is a function of Mach number and Reynolds number. The dependence of the latter can be neglected and thus P may be regarded as a function of Mach number rather than the velocity v in order to consider the effect of variation of velocity of sound. Hence $P(V)$ should be read as a function of W written in the form[2]

$$W = \left(\frac{T_0}{T(y)}\right)^{\frac{1}{2}} V \tag{11}$$

in order to enter the correct Mach number. Here $T(y)$, the absolute temperature at an altitude y, is approximated by a simple function of y in the form

$$T(y) = T_0 - K_0\, y. \tag{12}$$

T_0 is the absolute temperature under the Standard Ballistic condition and is equal to 287.5° K and K_0 is the constant temperature gradient of the order of 0.00103406552.

The graph of $P(W)$ values from the 1940 Resistance Law Tables is plotted in Fig. 1. The retardation coefficient $P(W)$ remains almost constant and is taken as 0.444 for Mach number below one.

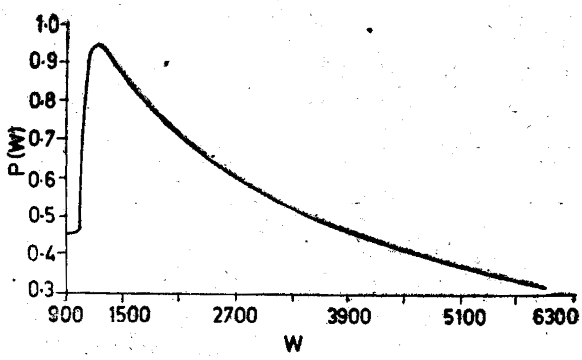

Fig. 1— Curve of 1940 Resistance Law.

The graph of $P(W)$ suggests the approximation of such values by second degree polynomial in the range $1270 < W \leqslant 6000$. Here the approximated functions for $W \leqslant 1270$ is considered the same as derived by Ghosh and Srinivasan[2] with the error everywhere less than 0.2 per cent from the correct value and mentioned in (1) and (2) for computational purposes.

The range $1270 < W \leqslant 6000$ is divided into three parts for better approximation with second degree polynomial in each part. Besides, a better approximation other than that obtained in Eqn. (3) has also been obtained for the range $1270 < W \leqslant 3000$ which has minimised the error everywhere less than 0.15 per cent from the correct value.

With these approximations for 1940 Resistance Law Tables the error for all practical purposes was everywhere less than 0.20 per cent except in the range $4640 \leqslant W \leqslant 4700$ where it is of the order of 0.32 per cent from the correct value.

EQUATIONS OF MOTION OF A PROJECTILE

The equations of motion[3,4] of an artillery shell with simplified assumptions that the projectile moves with its axis tangential to its path under two forces only viz., tangential retarding force due to air resistance and constant gravitational force, can be easily reduced to the form as shown in Fig. 2.

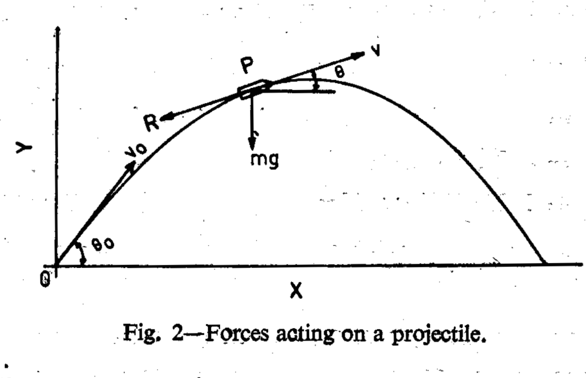

Fig. 2—Forces acting on a projectile.

$$\dot{x} = u \tag{13}$$

$$\dot{y} = up \tag{14}$$

$$\dot{p} = -g/u \tag{15}$$

$$\dot{u} = -\frac{1}{10^4 C_0 f(y)} u^2 \sqrt{1+p^2}\, P(W) \tag{16}$$

where

u denotes the horizontal component of velocity of the projectile.

p the tangent of the angle between the velocity vector and the x — axis.

$$W = v\sqrt{\frac{T_0}{T(y)}} = u\sqrt{1+p^2}\sqrt{\frac{T_0}{T(y)}}$$

APPROXIMATING $P(W)$ BY FUNCTION OR FUNCTIONS

The graph of the retardation coefficient $P(W)$ values (Fig. 1) suggests the approximation of P by a suitable second degree polynomial in the range $1270 < W \leqslant 6000$. The approximated function[2] for $W \leqslant 1270$, gave better results and hence the same are also considered here for computational purposes. The succeeding paragraphs explain the scheme for fitting the retardation coefficient in the range $1270 < W \leqslant 6000$.

Approximating P (W) by a function

On fitting the curve of second degree polynomial of the form

$$P(W) = A + BW + CW^2, \quad \text{for } 1270 < W \leqslant 6000 \tag{17}$$

by the method of least squares and after solving the normal equations, the constants come out to be

$$\left.\begin{aligned} A &= 1.286209 \\ B &= -0.320358 \times 10^{-3} \\ C &= 0.0272258 \times 10^{-3} \end{aligned}\right\} \tag{18}$$

The approximated function is not satisfactory as the percentage error is of the order of 6.512 per cent from the correct value. The analysis suggests the necessity of approximating by more functions which may yield better results.

Approximating P (W) by two functions

Feasibility of approximating $P(W)$ by second degree polynomial separately for two suitable velocity intervals has been studied on the basis of the trend of $P(W)$ curve and the following were observed.

$$\left.\begin{aligned} &\text{For } 1270 < W \leqslant 5000,\ P(W) = A_1 + B_1 W + C_1 W^2 \\ &\text{For } 5000 < W \leqslant 6000,\ P(W) = A_2 + B_2 W + C_2 W^2 \end{aligned}\right\} \tag{19}$$

The constants are worked out as

$$\left.\begin{aligned} A_1 &= 1.333576 & A_2 &= 0.852 \\ B_1' &= -0.357682 \times 10^{-3} & B_2 &= -0.129 \times 10^{-3} \\ C_1 &= 0.0336241 \times 10^{-6} & C_2 &= 0.0073 \times 10^{-6} \end{aligned}\right\} \tag{20}$$

The error with these approximations is of the order of 3.282 and 5.61 per cent in the two respective ranges. To achieve still better results, the range of the first interval is decreased as :

$$\left.\begin{aligned} &\text{For } 1270 < W \leqslant 4700,\ P(W) = A_1 + B_1 W + C_1 W^2 \\ &\text{For } 4700 < W \leqslant 6000,\ P(W) = A_2 + B_2 W + C_2 W^2 \end{aligned}\right\} \tag{21}$$

The constants are

$$\left.\begin{aligned} A_1 &= 1.346828 & A_2 &= 0.889 \\ B_1 &= -0.368905 \times 10^{-3} & B_2 &= -0.1466 \times 10^{-3} \\ C_1 &= 0.0357299 \times 10^{-6} & C_2 &= 0.00872 \times 10^{-6} \end{aligned}\right\} \tag{22}$$

The percentage error on the basis of these approximations was everywhere less than 2.5 and 0.20 per cent in their respective ranges from the correct value. The interval is further shortened and the constants for the second degree polynomial in the new ranges obtained as

$$\left.\begin{array}{ll} \text{For } 1270 < W \leqslant 4500 & \text{For } 4500 < W \leqslant 6000 \\ A_1 = 1.355579 & A_2 = 0.9067 \\ B_1 = -0.376538 \times 10^{-3} & B_2 = -0.1527 \times 10^{-3} \\ C_1 = 0.0372175 \times 10^{-6} & C_2 = 0.00925 \times 10^{-6} \end{array}\right\} \tag{23}$$

With these approximations, the percentage error was everywhere less than 2.0 and 0.26 per cent from the correct value separately in the two ranges. To improve the results, the interval of range is finally decreased to 914 m/sec. and the second degree polynomial curve fitted separately for each part of the ranges.

$$\left.\begin{array}{ll} \text{For } 1270 < W \leqslant 3000, & P(W) = A_1 + B_1 W + C_1 W^2 \\ \text{For } 3000 < W \leqslant 6000, & P(W) = A_2 + B_2 W + C_2 W^2 \end{array}\right\} \tag{24}$$

The constants are worked out as

$$\left.\begin{array}{ll} A_1 = 1.3972 & A_2 = 1.06906 \\ B_1 = -0.41654 \times 10^{-3} & B_2 = -0.21781 \times 10^{-3} \\ C_1 = 0.046088 \times 10^{-3} & C_2 = 0.01567 \times 10^{-6} \end{array}\right\} \tag{25}$$

The percentage error in the range $1270 < W \leqslant 3000$ is everywhere less than 0.15 per cent as compared to 0.16 per cent obtained by Ghosh and Srinivasan[2]. This shows a better approximation than that worked out earlier by Ghosh and Srinivasan for this particular range. But the error is of the order of 1.092 per cent in the range $3000 < W \leqslant 6000$ from the correct value. It is therefore sought to break up this second interval into further two ranges for further improvement in the approximated values of $P(W)$.

Approximating $P(W)$ by three functions

The approximation of the 1940 Resistance Law Tables by second degree polynomial separately in each part of the three valid ranges $1270 < W \leqslant 3000$, $3000 < W \leqslant 4700$, and $4700 < W \leqslant 6000$ is quite accurate, the error being less than 0.20 per cent except for the range $4640 \leqslant W \leqslant 4700$ where it is of the order of 0.32 per cent. The approximated function $P(W)$ is defined as

$$P(W) = \left\{\begin{array}{ll} A_1 + B_1 W + C_1 W^2, & \text{for } 1270 < W \leqslant 3000 \\ A_2 + B_2 W + C_2 W^2, & \text{for } 3000 < W \leqslant 4700 \\ A_3 + B_3 W + C_3 W^2, & \text{for } 4700 < W \leqslant 6000 \end{array}\right\} \tag{26}$$

where the values of the constants are

$$\left.\begin{array}{lll} A_1 = 1.3972 & A_2 = 1.169 & A_3 = 0.889 \\ B_1 = -0.41654 \times 10^{-3} & B_2 = -0.2697 \times 10^{-3} & B_3 = -0.1466 \times 10^{-3} \\ C_1 = 0.046088 \times 10^{-6} & C_2 = 0.02229 \times 10^{-6} & C_3 = 0.00872 \times 10^{-6} \end{array}\right\} \tag{27}$$

The analysis also reflects the best approximations in these three defined valid ranges.

Finally, the 1940 Resistance Law Tables for the standard projectiles moving in air are approximated from (1), (2), (4), (5), (26) and (27) by suitable functions separately for each part valid in the five different ranges with better results and are defined as

$$
P(W) = \left\{ \begin{array}{l}
0.444, \text{ for } W < 1000 \\
a_0 + \sum_{n=1}^{4} \left(a_n \cos nx + b_n \sin nx \right), \text{ for } 1000 \leqslant W \leqslant 1210, \\
\text{where } x = \left(\frac{W - 10^3}{200} \right) \pi \\
d_0 + d_1 W, \text{ for } 1210 < W \leqslant 1270 \\
A_1 + B_1 W + C_1 W^2, \text{ for } 1270 < W \leqslant 3000 \\
A_2 + B_2 W + C_2 W^2, \text{ for } 3000 < W \leqslant 4700 \\
A_3 + B_3 W + C_3 W^2, \text{ for } 4700 < W \leqslant 6000
\end{array} \right\} \quad (28)
$$

The constants on the basis of curve fitting by the method of least squares are

$$
\left. \begin{array}{ll}
a_0 = +0.769737 & b_1 = -0.088167 \\
a_1 = -0.274767 & b_2 = -0.012182 \\
a_2 = -0.090770 & b_3 = +0.029389 \\
a_3 = +0.021537 & b_4 = +0.006091 \\
a_4 = +0.018603 &
\end{array} \right\} \quad (29)
$$

$$
\left. \begin{array}{lll}
A_1 = 1.3972 & A_2 = 1.169 & A_3 = 0.889 \\
B_1 = -0.41654 \times 10^{-3} & B_2 = -0.2697 \times 10^{-3} & B_3 = -0.1466 \times 10^{-3} \\
C_1 = 0.046088 \times 10^{-6} & C_2 = -0.02229 \times 10^{-6} & C_3 = 0.00872 \times 10^{-6}
\end{array} \right\} \quad (30)
$$

The constants for the straightline approximation are

$$
\left. \begin{array}{l}
d_0 = 1.17736 \\
d_1 = -0.186 \times 10^{-3}
\end{array} \right\} \quad (31)
$$

These approximations involve 20 constants and the error is everywhere less than 0.20 per cent except for the case $4640 \leqslant W \leqslant 4700$ where it is of the order of 0.32 per cent.

NUMERICAL INTEGRATION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

The system of differential Eqns. (13) to (16) is numerically integrated step by step with the help of Milne's corrected formula[5,6]. The initial conditions of the Eqns. of motion are

$$
t_0 = 0,\ x_0 = 0,\ y_0 = 0,\ p_0 = \tan\theta_0\ \dot{x}_0 = v_0 \cos\theta_0 = u_0,\ \dot{p}_0 = -g/u_0,
$$

$$
\dot{y}_0 = \dot{x}_0\, p_0 = u_0\, p_0,\ \dot{u}_0 = -\frac{1}{10^4\, C_0 f(y_0)}\, u_0^2 \sqrt{1 + p_0^2}\ P(W) \quad (32)
$$

where

$$
W = u_0 \sqrt{1 + p_0^2} \sqrt{\frac{T_0}{T(y_0)}}
$$

The first two initial steps to use the Milne's formula for determining the elements of trajectory at equal interval $\triangle t$ have been computed as

$$
t_1 = t_0 + \triangle t
$$

$$
u_a = \dot{x}_0 + \triangle t\ \dot{u}_0,\quad p_a = p_0 + \triangle t\ \dot{p}_0,\quad v_a = u_a \sqrt{1 + p_a^2},\ W = v_a \sqrt{\frac{T_0}{T(y_0)}}
$$

where suffix a for approximate values.

$$\dot{u}_1 = -\frac{1}{10^4 C_0 f(y_0)} u_a^2 \sqrt{1 + p_a^2}\; P(W)$$

$$x_1 = \dot{x}_0 + \Delta t \frac{\dot{u}_0 + \dot{u}_1}{2}$$

$$x_1 = x_0 + \Delta t \frac{\dot{x}_0 + \dot{x}_1}{2}$$

$$\dot{p}_1 = -\frac{32.16}{\dot{x}_1}$$

$$p_1 = p_0 + \Delta t \frac{\dot{p}_0 + \dot{p}_1}{2}$$

$$\dot{y}_1 = \dot{x}_1 p_1$$

$$y_1 = y_0 + \Delta t \frac{\dot{y}_0 + \dot{y}_1}{2}$$

$$\theta_1 = \tan^{-1}(p_1)$$

For the second step

$t_2 = t_1 + \Delta t$ and so on.

The subsequent elements of trajectory at equal interval are computed on the basis of the Milne's formula

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1} \right)$$

where

h is the small interval

For $t_3 = t_2 + \Delta t$

$$u_a = \dot{x}_1 + 2\Delta t\, \dot{u}_1, \qquad p_a = p_1 + 2\Delta t\, \dot{p}_1, \; v_a = u_a\sqrt{1 + p^2 a}$$

$$W = v_a \sqrt{\frac{T_0}{T(y_2)}}$$

$$\dot{u}_3 = -\frac{1}{10^4 C_0 f(y_2)} u^2_a \sqrt{1 + p^2_a}\, P(W)$$

$$\dot{x}_3 = \dot{x}_1 + \frac{\Delta t}{3} (\dot{u}_1 + 4\,\dot{u}_2 + \dot{u}_3)$$

$$x_3 = x_1 + \frac{\Delta t}{3} (\dot{x}_1 + 4\dot{x}_2 + \dot{x}_3)$$

$$\dot{p}_3 = -\frac{32.16}{\dot{x}_3}$$

$$p_3 = p_1 + \frac{\Delta t}{3} (\dot{p}_1 + 4\dot{p}_2 + \dot{p}_3)$$

$$\dot{y}_3 = \dot{x}_3 p_3$$

$$y_3 = y_1 + \frac{\Delta t}{3} (\dot{y}_1 + 4\dot{y}_2 + \dot{y}_3)$$

$$\theta_3 = \tan^{-1}(p_3)$$

For the next interval

$t_4 = t_3 + \Delta t$ and so on.

$P(W)$ corresponding to each W were calculated from the approximated functions of retardation coefficient based on 1940 law from (28) to (31).

RESULTS

The trajectories of the projectiles for various angle of projections at velocities 640 m/sec and 1250 m/sec have been worked out by using the approximated functions $P(W)$ and are shown in Fig. 3 & 4.

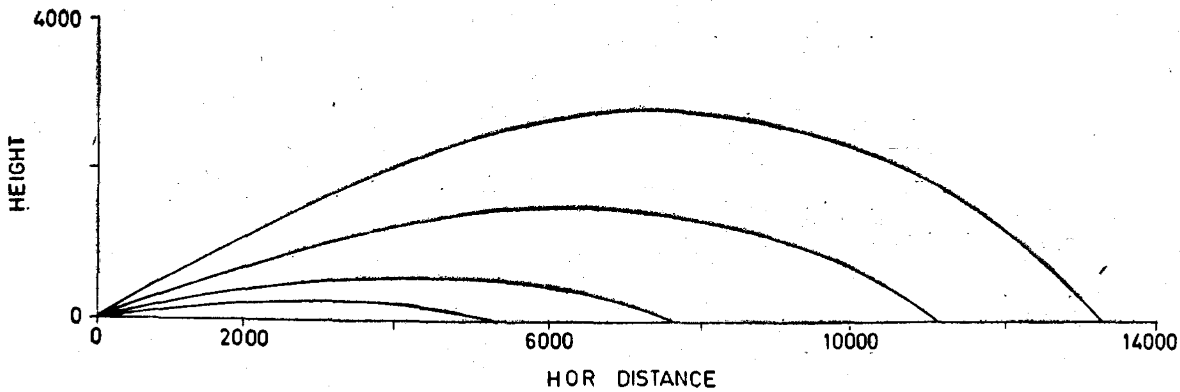

Fig. 3—Trajectories—distance vs height; velocity=640 m/sec, C_0=2.0; angle of projection—5°, 10°, 20° & 30°.

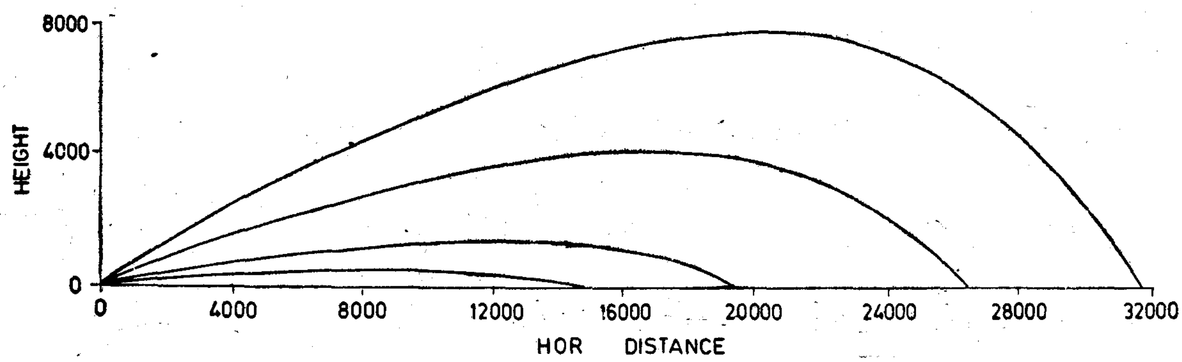

Fig. 4—Trajectories—distance vs height; velocity=1250 m/sec, C_0=2.5; angle of projection—5°, 10°, 20° & 30°.

The trajectory results for velocity 640 m/sec are comparable with those obtained from the Siacci's method using Ballistic functions and the associated values. Due to certain limitations of using Ballistic functions and the associated values the results for velocity 1250 m/sec cannot be compared.

However, for lower velocities this method has shown an agreement to a great extent with the values obtained by the Siacci's method. Due to inherent difficulties in applying Siacci's method for a range beyond 914 m/sec, it is suggested that the method given in this paper be applied in case of trajectories of projectiles for higher velocities (up to 1829 m/sec). Further it is clarified that this method can also be adopted for lower velocities as well.

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REFERENCES

1. Bliss, Gilbert, A., 'Mathematics for Exterior Ballistics' (John Wiley & Sons, Inc.), 1944, p. 27.
2. Ghosh, A., & Srinivasan, B., *Sankhya*, *Ind. J. of Stat., Series B*, **32**, Parts 3&4 (1970), 363-378.
3. 'Lecture Notes on Ballistics' (Defence Science Laboratory, Delhi), 1952, p. 192.
4. McShane, E.J., Kelly, J.L. & Reno, F.V., 'Exterior Ballistics' (University of Denver Press), 1953, p. 192.
5. Scarborough, J.B., 'Numerical Mathematical Analysis' (John Hopkins Press), 1966, p. 375.
6. Gerald, Curtis, F., 'Applied Numerical Analysis' (Addison-Wesley Publishing Co. Massachussetts), 1970, p. 119.