STEADY FLOW OF MICROPOLAR INCOMPRESSIBLE FLUID BETWEEN TWO PARALLEL POROUS PLATES

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Steady flow of a micropolar incompressible fluid between two parallel porous plates $y = 0$ and $y = h$ is studied by perturbation method. Injection velocity at $y=0$ is V_1 and at $y=h$ the suction velocity is V_2. The behaviour of the various flow variables is investigated for varying values of the micropolarity parameter K/μ. It is observed that the longitudinal velocity is more at the lower plate and less at the upper plate than in the non-polar case. It is also observed the wall friction is more at both plates when compared to the non-polar case.

The study of micropolar fluid was initiated by Eringen[1]. These fluids exhibit certain microscopic effects, which arise from the local structure and micromotion of fluid elements. The fluids experience couple stresses and the stress tensor has antisymmetric components. An independent kinematic vector called micro-rotation is introduced and one has to solve two symultaneous equations in the velocity vector and the micro-rotation vector. Some problems of practical interest were investigated by Lakshmana Rao[2,3,4]. In the present paper the study of such a fluid between two parallel porous plates is investigated. In the cartesian co-ordinate system (x, y, z) the main flow is in the z-direction bounded by porous plates $y = 0$ and $y=h$. The injection velocity at $y=0$ is V_1 and at $y=h$ the suction velocity is V_2. The problem is solved by the method of parameter perturbation, the parameter being the suction Reynolds number $R_2 = \frac{\rho h V_2}{\mu}$. The behaviour of various flow variables is discussed for different values of the micropolarity parameter $\frac{K}{\mu}$.

EQUATIONS OF MOTION

The governing equations in the absence of body force and body couple are[1]

$$\rho \frac{d\vec{q}}{dt} = - \operatorname{grad} P + K \operatorname{curl} \vec{\nu} + (\mu + K) \nabla^2 \vec{q} \tag{1}$$

$$\rho j \frac{d\vec{\nu}}{dt} = -2K \vec{\nu} + K \operatorname{curl} \vec{q} - \gamma \operatorname{curl} \operatorname{curl} \vec{\nu} + (\alpha + \beta + \gamma) \operatorname{grad} \operatorname{div} \vec{\nu} \tag{2}$$

and the equation of continuity

$$\operatorname{div} \vec{q} = 0 \tag{3}$$

In the above equations $\vec{q}$, $\vec{\nu}$ are respectively the velocity and micro-rotation vectors and P is the fluid pressure. ρ and j are the fluid density and microgyration parameter, and $\alpha, \beta, \gamma, K,$ and μ are viscosity coefficients which are taken to be constant in the present investigation. The stress tensor t_{ij} and the couple stress tensor m_{ij} are given by[2]

$$t_{ij} = - P\delta_{ij} + (2\mu + K)\, e_{ij} + K\epsilon_{ijm} (\omega_m - \nu_m) \tag{4}$$

$$m_{ij} = \alpha \nu_{K,K}\, \delta_{ij} + \beta \nu_{i,j} + \gamma \nu_{j,i} \tag{5}$$

where $\vec{\omega}$ is the vorticity vector. δ_{ij} is the Kronecker delta and ϵ_{ijm} is the alternating symbol.

In the present case the main flow is in z-direction bounded by the plates $y=0$ and $y=h$. All the flow variables are independent of the coordinate x. With $\vec{q}=(0, V, u)$ the vorticity vector has only the x-component and therefore we take $\vec{\nu} = C_i$, where i is the unit vector in the x-direction[2]. With this we observe that div $\vec{\nu}=0$ and the last term in Eq. (2) vanishes. With $\eta=\frac{y}{h}$, the boundary conditions are the usual adherence condition.

$$\left.\begin{array}{l} u(z,0)=u(z,1)=0 \\ C(z,0)=C(z,1)=0 \end{array}\right\} \quad (6)$$

and for V, we have

$$V(z,0)=V_1,\ V(z,1)=V_2 \quad (7)$$

Now following Terill & Shrestha[7], we take the stream function $\psi(z,\eta)$ in the form

$$\psi(z,\eta)=\left(\frac{h\bar{U}}{\alpha_2}-V_2\frac{z}{h}\right)f(\eta) \quad (8)$$

where $\bar{U}$ is the entrance velocity and $\alpha_2 = 1-\frac{V_1}{V_2}$. The Eq. (3) is automatically satisfied and the velocity components u and V are given by

$$\left.\begin{array}{l} u=\left(\frac{\bar{U}}{\alpha_2}-V_2\frac{z}{h}\right)f'(\eta) \\ V=V_2 f(\eta) \end{array}\right\} \quad (9)$$

we also take

$$C=\left(\frac{\bar{U}}{h\alpha_2}-V_2\frac{z}{h^2}\right)g(\eta) \quad (10)$$

where $f(\eta)$ and $g(\eta)$ are functions of η to be determined. The equations of motion (1) now give

$$\left(1+\frac{K}{\mu}\right)\left\{R_2 f''-R_2 S_2 g\right\}-R_2^2 ff'=S_1\frac{\partial P}{\partial\eta}\left(1+\frac{K}{\mu}\right)^2 \quad (11)$$

$$\left(1+\frac{K}{\mu}\right)\left\{f'''-S_2 g'\right\}+R_2\left\{f'^2-ff''\right\}=\frac{\partial P}{\partial z}\frac{h^3\alpha_2}{\mu(h\bar{U}-V_2\alpha_2 z)} \quad (12)$$

eliminating the pressure between the above two equations, we have

$$\left(1+\frac{K}{\mu}\right)\left\{f'''-S_2 g'\right\}+R_2\left\{f'^2-ff''\right\}=M\left(1+\frac{K}{\mu}\right) \quad (13)$$

where M is a constant to be determined.

Eq. (2) using (10) gives

$$\left(1+\frac{K}{\mu}\right)\left\{S_3 g''+S_2 f''-2S_2 g\right\}=R_2 J\left(fg'-gf'\right) \quad (14)$$

In the above equations (11) to (14) dashes denote differentiation with respect to η and

$$R_2=\frac{\rho h V_2}{\mu},\quad S_1=\frac{\rho h^2}{(\mu+K)^2},\quad S_2=\frac{K}{(\mu+K)}$$

$$S_3=\frac{\gamma}{h^2(\mu+K)} \quad \text{and} \quad J=\frac{j}{h^2}$$

In terms of $f(\eta)$ and $g(\eta)$ the boundary conditions (6) to (7) give

$$\left.\begin{array}{ll} f'(0)=f'(1)=0 & g(0)=g(1)=0 \\ f(0)=1-\alpha_2 & f(1)=1 \end{array}\right\} \quad (15)$$

The complete solution of the problem consists of solving (13 to (14) along with (15). Eq. (12) will then give the pressure gradient in the z-direction.

PERTURBATION SOLUTION FOR SMALL R_2

We take the following series solutions for $f(\eta)$, $g(\eta)$ and the constant M

$$\left.\begin{aligned} f(\eta) &= \sum_{r=0}^{\infty} R_2^r f_r(\eta)\,,\; g(\eta) = \sum_{r=0}^{\infty} R_2^r g_r(\eta) \\ M &= \sum_{r=0}^{\infty} M_r R_2^r \end{aligned}\right\} \tag{16}$$

with the boundary conditions

$$\left.\begin{aligned} & f_r'(0) = f_r'(1) = 0\,,\; g_r(0) = g_r(1) = 0 \qquad r \geqslant 0 \\ & f_0(0) = 1 - \alpha_2\,,\; f_0(1) = 1 \text{ and} \\ & f_r(0) = f_r(1) = 0 \qquad r \geqslant 1 \end{aligned}\right\} \tag{17}$$

Substituting (16) in (13) and (14) and equating the various coefficients of powers of R_2, we get

Zero Order Approximation

$$f_0^{v} + \frac{S_2(S_2-2)}{S_3} f_0''' = -2\frac{S_2}{S_3} M_0 \tag{18}$$

$$2S_2^2 g_0 = S_3 f_0'^{v} + S_2^2 f_0'' \tag{19}$$

$$f_0'(0) = f_0'(1) = 0\,,\; g_0(0) = 0 = g_0(1)\,,\; f_0(0) = 1 - \alpha_2\,,\; f_0(1) = 1 \tag{20}$$

First Order Approximation

$$f_1^{v} + \frac{S_2(S_2-2)}{S_3} f_1''' = -2\frac{S_2}{S_3} M_1 + \left\{ 2\frac{S_2}{S_3}(f_0'^2 - f_0 f_0'') + J\frac{S_2}{S_3}(f_0 g_0'' - g_0 f_0'') + f_0 f_0'^{v} - f_0''^2 \right\} \Big/ \left(1 + \frac{K}{\mu}\right) \tag{21}$$

$$2S_2^2 g_1 = S_3 f_1'^{v} + S_2^2 f_1'' - \frac{J S_2}{\left(1+\frac{K}{\mu}\right)}(f_0 g_0' + g_0 f_0') + \frac{S_3}{\left(1+\frac{K}{\mu}\right)}(f_0' f_0'' - f_0 f_0''') \tag{22}$$

with $$f_1'(0) = f_1'(1) = 0\,,\; g_1(0) = g_1(1) = 0\,,\; f_1(0) = f_1(1) = 0 \tag{23}$$

from (18) to (23), we find

$$f_0 = E_0 + D_0\eta + A_0\eta^2 + \frac{M_0\eta^3}{3(2-S_2)} + \frac{S_3 B_0 e^{m\eta}}{(2-S_2)} + \frac{S_3 C_0 e^{-m\eta}}{(2-S_2)}$$

where

$$m^2 = \frac{(2-S_2)S_2}{S_3}$$

$$f_1 = \Big[E1 + D1\eta + A4\eta^2/2 + B4\eta^3/6 + C4\eta^4/12 + D4\eta^5/20 + E4\eta^6/30 +$$
$$+ F4\,\eta^7/42 + e^{m\eta} \Big\{ m^4 G4 - 2m^3 H4 + 6m^2 N4 - 24m\, D4 + 120\, P4 \Big\} \Big/ m^6 +$$
$$+ \eta e^{m\eta} \Big\{ m^3 H4 - 4m^2 N4 + 18m\, D4 - 96P4 \Big\} \Big/ \eta^5 + \eta^2 e^{m\eta} \Big\{ m^2 N4 -$$
$$- 6m\, D4 + 36P4 \Big\} \Big/ m^4 + \eta^3 e^{m\eta} \Big\{ m\, D4 - 8P4 \Big\} \Big/ m^3 + \eta^4 e^{m\eta} P4/m^2 +$$
$$+ e^{-m\eta} \Big\{ m^4 Q4 + 2m^3 R4 + 6m^2 S4 + 24m\, T4 + 120\, U4 \Big\} \Big/ m^6 + \eta e^{-m\eta} \cdot$$
$$\cdot \Big\{ m^3 R4 + 4m^2 S4 + 18m\, T4 + 96\, U4 \Big\} \Big/ m^5 + \eta^2 e^{-m\eta} \Big\{ m^2 S4 +$$
$$+ 6m\, T4 + 36\, U4 \Big\} \Big/ m^4 + \eta^3 e^{-m\eta} \Big\{ mT4 + 8U4 \Big\} \Big/ m^3 + \eta^4 e^{-m\eta} \cdot$$
$$\cdot \Big\{ U4/m^2 \Big\} \Big] \left(\frac{\mu}{K+\mu} \right)$$

The expressions of the constants are omitted; since they are very lengthy.

LONGITUDINAL VELOCITY FIELD

From (9) the axial velocity in the non-dimensional form is

$$<u> = \frac{u}{\bar{U}} = \left(\frac{1}{\alpha_2} - \frac{R_2}{R} \frac{z}{h} \right) \left(f_0' + R_2 f'_1 + \ldots \right) \qquad (24)$$

where $R_2 = \dfrac{\rho h \bar{U}}{\mu}$

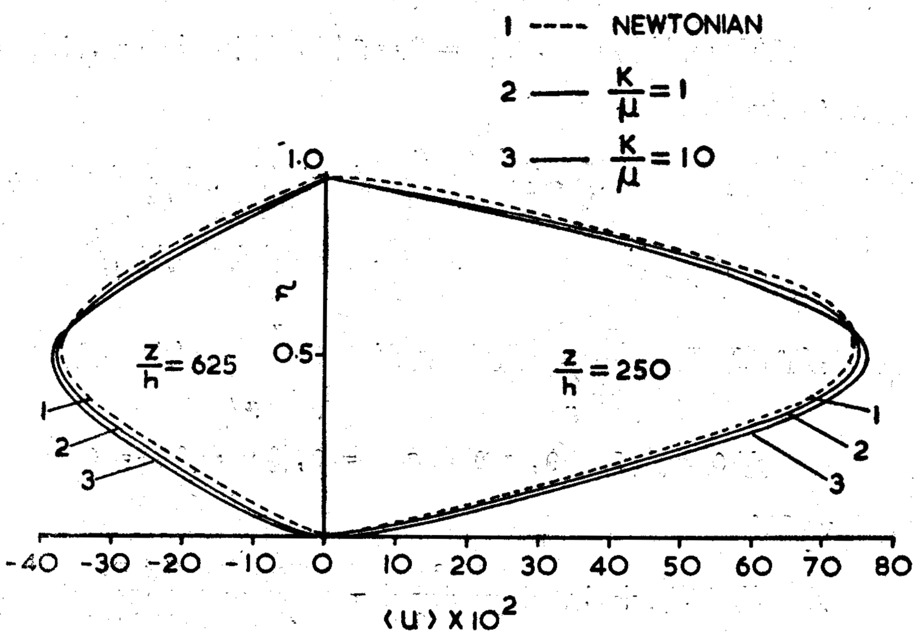

Fig. 1—Longitudinal velocity profile $<u>$.

In the Fig. 1 the longitudinal velocity profiles for

$$R_2 = 0{\cdot}8\,,\ \alpha_2 = 0{\cdot}5\,,\ R = 200\,,\ \text{and}\ S_3 = 0{\cdot}1875$$

for different values of $\frac{K}{\mu}$ are drawn. In the present problem we have assumed that the micro-rotation vector is zero on both the plates $\eta = 0$ and $\eta = 1$. Therefore the only physical quantity that effects the flow in present case in addition to the ones that are there in the classical case is the couple stress. It is known that for one-dimensional flows between the solid boundaries the effect of the couple stress is to decrease the longitudinal velocity in the entire flow region[5]. But in the present case the couple stress increases the velocity at the lower plate through which the fluid is being injected in the flow region. At the upper plate the fluid is being sucked out and here the couple stress decreases the velocity. This is true whether the longitudinal velocity is positive, near the entry or negative far away from the entry. We also observe from the figure that the maximum value of $<u>$ moves towards the lower plate as $\frac{K}{\mu}$ increases.

PRESSURE FIELD

The pressure drop in the y and z directions in the non-dimensional form are

$$<P_{ij}> = \frac{\{P(z,0) - P(z,\eta)\} h}{\mu \bar{U}} = \frac{R_2 h}{\bar{U}(\mu + K)} \left\{ - A_0 \frac{(2 - S_2)\eta}{S_1} - \frac{M_0 \eta^2}{2 S_1} \right\} \qquad (25)$$

$$<P_z> = \frac{\{P(0,\eta) - P(z,\eta)\} h}{\mu \bar{U}} = h(M_0 + R_2 M_1)\left(1 + \frac{K}{\mu}\right)\left\{ \frac{V_2}{2\bar{U}} \left(\frac{z}{h}\right)^2 - \frac{1}{\alpha_2}\frac{z}{h} \right\} \qquad (26)$$

Both are parabolic in their respective coordinates. Graphs of $<P_z>$ are drawn in Fig. 2 for different

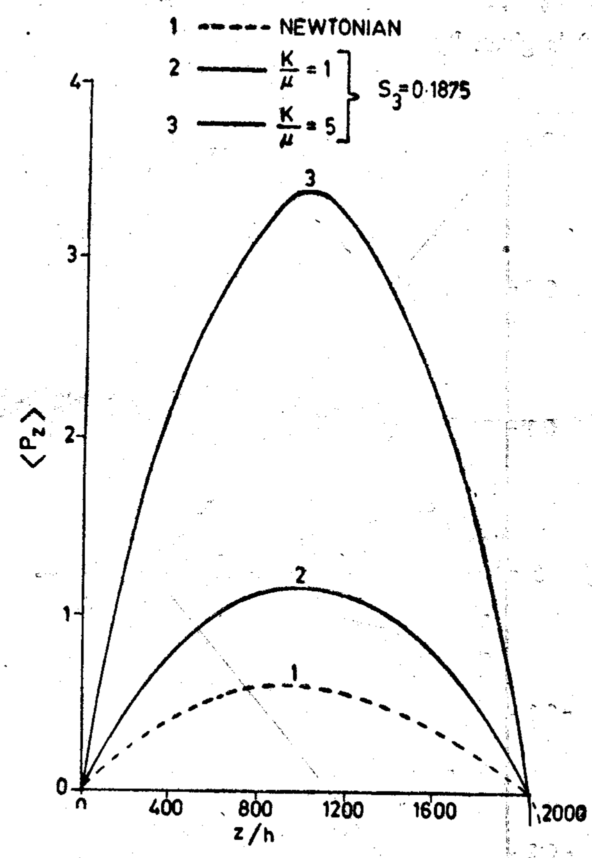

Fig. 2—Axial pressure difference $<P_z>$.

values of $\frac{K}{\mu}$, we observe that with increasing values of $\frac{K}{\mu}$, $<P_z>$ increases. It is positive near the entry and for large values of $\frac{z}{h}$ it becomes negative. As in the classical case the axial pressure gradient $<P_z>$

does not depend on η $<P_z>$ is more pronounced in the present than in the classical case[6].

STRESS COMPONENTS

The non-vanishing stress components in the non-dimensional form are

$$T_{xx} = \frac{t_{xx}}{P} = -1 \tag{27}$$

$$T_{yy} = \frac{t_{yy}\, h}{(2\mu + K)V_2} = -\frac{Ph}{(2\mu + K)V_2} + f' \tag{28}$$

$$T_{zz} = \frac{t_{zz}\, h}{(2\mu + K)V_2} = -\frac{Ph}{(2\mu + K)V_2} - f' \tag{29}$$

$$T_{yz} = \frac{t_{yz}}{\left(\frac{U}{h\alpha_2} - \frac{V_2 z}{h}\right)\mu} = \left(1 + \frac{K}{\mu}\right) f'' - \frac{K}{\mu}\, g \tag{30}$$

$$T_{zy} = \frac{t_{zy}}{\left(\frac{U}{h\alpha_2} - \frac{V_2 z}{h}\right)\mu} = f'' + \frac{K}{\mu}\, g \tag{31}$$

In the present case the boundaries being $\eta = 0$ and $\eta = 1$, the coefficient of skin friction C_F on these plates are given by

$$C_F = \left(\frac{2t_{yz}}{\rho U^2}\right)_{\eta=0,\,1} = \frac{2}{R}\left\{\frac{1}{\alpha_2} - \frac{R_2 z}{R h}\right\}\left\{\left(1 + \frac{K}{\mu}\right) f'' - \frac{K}{\mu}\, g\right\}_{\eta=0,\,1} \tag{32}$$

The coefficient of $\frac{K}{\mu}$ in (30) is given by

$$T_{[yz]} = (f'' - g) \tag{33}$$

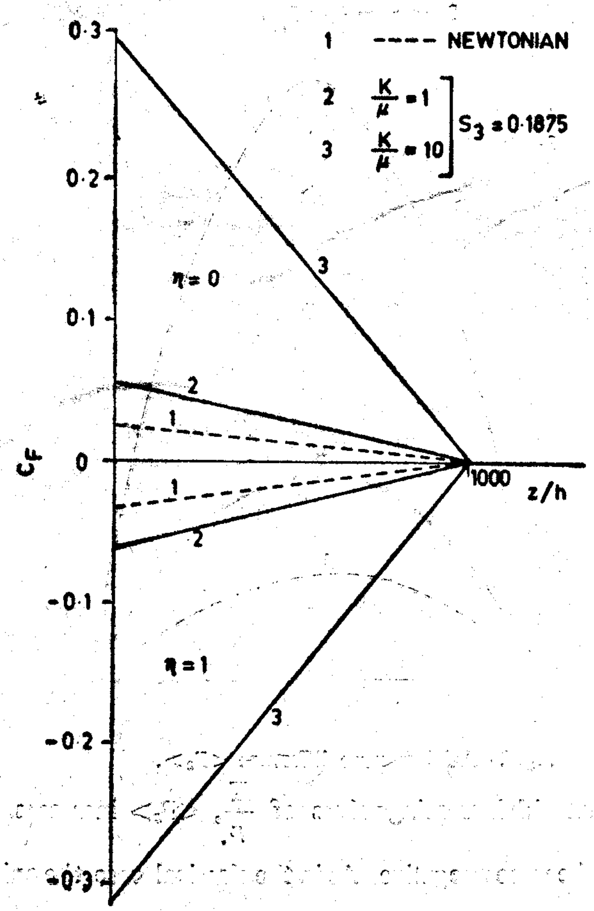

Fig. 3—Skin friction.

The graph of the coefficient of skin friction C_F at the both plates is drawn in Fig. 3 for various values of $\frac{K}{\mu}$. The coefficient of skin friction in the present case is more than that in the classical case[6] and this is due to the antisymmetric part given by (33). On the boundaries $g(\eta)$ is zero therefore C_F increases because of the vorticity term $f''(\eta)$. For any value of $\frac{K}{\mu}$ skin friction is more at the upper plate than at the lower plate.

MICRO ROTATION VECTOR AND COUPLE STRESSES

In the present problem the micro-rotation vector has only the component. In the non-dimensional form it is given by

$$<C> = \frac{Ch}{U} = \left(\frac{1}{a_2} - \frac{R_2}{R}\,\frac{z}{h}\right)\left(g_0 + R_2\, g_1\right) \tag{34}$$

and the non-vanishing couple stresses in non-dimensional form are given by

$$< m_{xy} > = \frac{m_{xy}}{\beta\left(\frac{U}{h^2\, a_2} - \frac{V_2 z}{h^3}\right)} = g' \tag{35}$$

$$< m_{xz} > = \frac{m_{xz}\, h^2}{\beta\, V_2} = -g \tag{36}$$

$$< m_{yx} > = \frac{m_{yx}}{\mu\left(\frac{U}{a_2} - V_2\,\frac{z}{h}\right)} = S_3\left(1 + \frac{K}{\mu}\right)g' \tag{37}$$

$$< m_{zx} > = \frac{m_{zx}}{\mu V_2} = -\,S_3\left(1 + \frac{K}{\mu}\right)g \tag{38}$$

In Fig. 4, the graph of $<C>$ is drawn. It is seen that near the entrance, it is positive near the plate $\eta = 0$

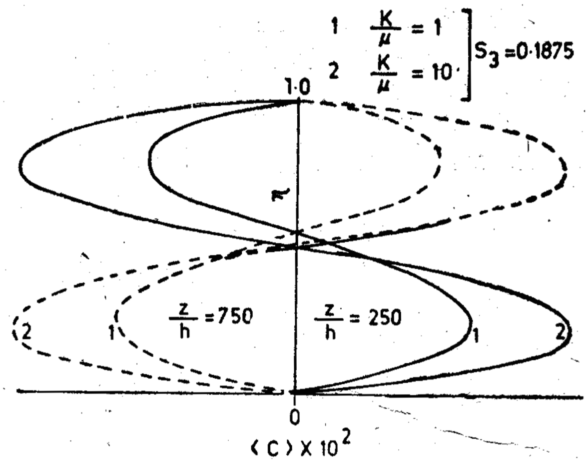

Fig. 4—Non-dimensional micro-rotation $<C>$.

and negative at the upper plate $\eta = 1$, the reverse happens for large values of $\frac{z}{h}$. In both cases it increases in magnitude with increasing $\frac{K}{\mu}$. The plate η is constant on which it is zero moves towards the lower plate with increasing $\frac{K}{\mu}$.

The boundaries being $y = 0$ and $y = h$, in this case the graph of $<m_{yx}>$ which act on these boundaries

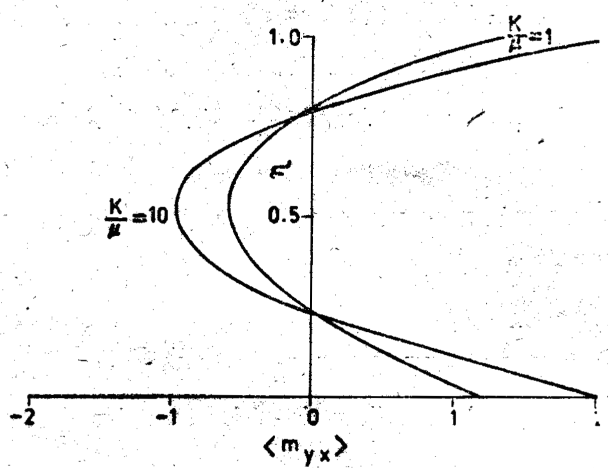

Fig. 5—Non-dimensional couple stress $<m_{yx}>$.

is drawn in Fig. 5. $<m_{yx}>$ is positive at both plates and is negative in the middle region increasing in magnitude with $\frac{K}{\mu}$. The couple stress $<m_{xz}>$ is independent of z and its behaviour with respect to η is just the reverse of $<C>$ as can be seen by comparing (34) and (36).

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