CERTAIN GENERATING FUNCTIONS

(KM.) SHOBHA SAXENA

Lucknow University, Lucknow

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This paper offers a brief account of the generating functions originated from a relation of the type

$$\sum_{n=0}^{\infty} \Psi_n t^n = (1-t)^{-\lambda} \cdot F\left[\begin{matrix} \lambda : \rho-\mu \,;\, \mu,\, (\alpha_p)\,; \\ \rho : \text{———}\,;\, (\beta_q)\,; \end{matrix} \frac{t}{t-1}, \frac{xt}{t-1}\right] \cdot \sum_{k=0}^{\infty} \phi_k t^k$$

where,

$$\Psi_n = \sum_{k=0}^{n} \frac{(\lambda)_{n-k}(\mu)_{n-k}}{(\rho)_{n-k}(n-k)!}\, {}_{p+1}F_q\left[\begin{matrix} -n+k,\, (\alpha_p)\,; \\ (\beta_q)\,; \end{matrix} x\right] . \Phi_k .$$

Here, $\sum_{n=0}^{\infty} \Psi_n t^n$ converges and Φ_k is a suitable sequence such that $\sum_{k=0}^{\infty} \phi_k t^k$ converges. Such generating functions are valuable in the study of polynomial sets and other special functions which occur in physical problems and as solutions of differential equations.

Recently a series of generating functions for ultraspherical polynomials have been given by Brafman[1], Brown[2] and Saxena[3]. Srivastava[4] and Varma[5] have also given double hypergeometric functions as generating functions for Jacobi and Laguerre polynomials. In this paper we have given certain more general generating functions for ultraspherical polynomials involving Kampé dé Feriét type of functions.

THEOREM

Let us consider a relation

$$\Psi_n = \sum_{k=0}^{n} \frac{(\lambda)_{n-k}(\mu)_{n-k}}{(\rho)_{n-k}(n-k)!} \cdot {}_{p+1}F_q\left[\begin{matrix} -n+k,\, (\alpha_p)\,; \\ (\beta_q)\,; \end{matrix} x\right] . \Phi_k \tag{1}$$

for any given sequence Φ_n $(n \geqslant 0)$, such that $\sum_{n=0}^{\infty} \phi_n t^n$ converges absolutely. Then

$$\sum_{n=0}^{\infty} \Psi_n t^n = (1-t)^{-\lambda} F\left[\begin{matrix} \lambda : \rho-\mu \,;\, \mu,\, (\alpha_p)\,; \\ \rho : \text{———}\,;\, (\beta_q)\,; \end{matrix} \frac{t}{t-1}, \frac{xt}{t-1}\right] \sum_{k=0}^{\infty} \phi_k t^k , \tag{2}$$

where (α_p) is written for $\alpha_1, \alpha_2 \ldots, \alpha_p$ and the series $\sum_{n=0}^{\infty} \Psi_n t^n$ converges.

The proof easily follows by using the result obtained by Srivastava[4]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{(\rho)_n}\, {}_{p+1}F_q\left[\begin{matrix} -n,\, (a_p)\,; \\ (b_q)\,; \end{matrix} x\right] \frac{t^n}{n!}$$

$$= (1-t)^{-\lambda} . F\left[\begin{matrix} \lambda : \rho-\mu \,;\, \mu\,,\, (a_p)\,; \\ \rho : \text{———}\,;\quad (b_q)\,; \end{matrix} \frac{t}{t-1}, \frac{xt}{t-1}\right] \tag{3}$$

which is a special case of (2) when $\Phi_0 = 1$ and $\Phi_n = 0$ for $n \geqslant 1$, and which is an extension of Varma's result[5]. The Kampé dé Feriét type of function

$$F\left(\begin{matrix}\mu \\ \nu \\ \rho \\ \sigma\end{matrix}\left|\begin{matrix}\alpha_1,\ldots,\alpha_\mu \\ \beta_1,\beta'_1,\ldots,\beta_\nu,\beta'_\nu \\ \gamma_1,\ldots,\gamma'_\rho \\ \delta_1,\delta'_1,\ldots,\delta_\sigma,\delta'_\sigma\end{matrix}\right| x, y\right) \equiv F\left[\begin{matrix}(\alpha\ ):(\beta_\nu),(\beta_{\nu'}\ ); \\ (\gamma_\rho\ ):(\delta_\sigma\ ),(\delta'_\sigma\ );\end{matrix} x, y\right]$$

is defined by the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j)_{m+n} \prod_{j=1}^{\nu}\{(\beta_j)_m(\beta'_j)_n\}}{\prod_{j=1}^{\rho}(\gamma_j)_{m+n} \prod_{j=1}^{\sigma}\{(\delta_j)_m(\delta'_j)_n\}} \cdot \frac{x^m y^n}{(1)_m(1)_n},$$

where $\mu+\nu \leqslant \rho+\sigma+1$.

PARTICULAR CASES

Case I

If in (1) and (2), we put $\lambda = \rho$, $p = q$, $(\alpha_p) = (\beta_p)$ and $\Phi_j = (\alpha)_j (1-X)^j/j!$, we get the generalized Jacobi polynomial

$$\Psi_n = \frac{(\mu)_n}{n!}(1-x)^n F\left(-n, \alpha; -\mu-n+1; \frac{X-1}{x-1}\right) \tag{4}$$

and its generating function

$$\sum_{n=0}^{\infty}\Psi_n t^n = (1-t)^{-\lambda} F_1\left(\lambda; \lambda-\mu, \mu; \lambda; \frac{t}{t-1}, \frac{xt}{t-1}\right) \cdot {}_1F_0(\alpha; —; (1-X)x)$$

$$= (1-t(1-x))^{-\mu}(1-t(1-X))^{-\alpha} \tag{5}$$

The first term in (5) was given by Saxena[3] and the next one by Carlitz[6]. (4) and (5) also extend a result by Niblett[7] and another due to Chaundy[8].

Case II

Putting

$$\Phi_j = \frac{(a)_j(b)_j}{(c)_j\, j!}\, {}_2F_1\left[\begin{matrix}a+j, b'; \\ c+j;\end{matrix}\ x\right]$$

in (1) and (2), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda)_{n-k}(\mu)_{n-k}(a)_k(b)_k}{(\rho)_{n-k}(c)_k(n-k)!\,k!}\, {}_{p+1}F_q\left[\begin{matrix}-n+k, (\alpha_p); \\ (\beta_q);\end{matrix}\ x\right] \cdot {}_2F_1\left[\begin{matrix}a+k, b'; \\ c+k;\end{matrix}\ x\right] t^n$$

$$= (1-t)^{-\lambda} F\left[\begin{matrix}\lambda:\rho-\mu;\mu, & (\alpha_p); \\ \rho: \text{———}; & (\beta_q);\end{matrix}\ \frac{t}{t-1}, \frac{xt}{t-1}\right] F^{(1)}\left[\begin{matrix}a;b,b'; \\ c;\end{matrix}\ x, t\right].$$

where $|x|+|t|<1$. (6)

For $p = q$, $\alpha_1 = \beta_1, \ldots, \alpha_p = \beta_p$, and $a = c$, the above reduces to

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\rho)_n\, n!}(1-x)^{n-b'}\, {}_3F_2\left[\begin{matrix}1-\rho-n, -n, b'; \\ 1-\lambda-n, 1-\mu-n;\end{matrix}\ \frac{1}{1-x}\right] t^n$$

$$= (1-t)^{-\lambda} F\left[\begin{matrix}\lambda:\rho-\mu;\mu; \\ \rho: \text{———};\end{matrix}\ \frac{t}{t-1}, \frac{xt}{t-1}\right] F^{(1)}[—; b, b'; —; x, t] \tag{7}$$

Case III

If, however, $$\Phi_j(x) = \frac{(a)_j\,(b)_j}{(c)_j\,j\,!}\;{}_2F_1\left[\begin{matrix} a+j\,,\,b\,; \\ c\,; \end{matrix}\; x\right],$$

(1) and (2) give the generating function

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda)_{n-k}\,(\mu)_{n-k}\,(a)_k\,(b)_k}{(\rho)_{n-k}\,(c)_k\,(n-k)\,!\,k\,!}\;{}_{p+1}F_q\left[\begin{matrix}-n+k\,,\,(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\;x\right]\cdot{}_2F_1\left[\begin{matrix}a+k\,,\,b\,; \\ c\,;\end{matrix}\;x\right]t^n$$

$$=(1-t)^{-\lambda}\,F\left[\begin{matrix}\lambda:\rho-\mu\,;\,\mu\,,\,(\alpha_p)\,; \\ \rho:\;\text{———};\;(\beta_q)\,;\end{matrix}\;\frac{t}{t-1}\,,\,\frac{xt}{t-1}\right]F^{(2)}\left[\begin{matrix}a\,,\,b\,,\,b'\,; \\ c\,,\,c'\,;\end{matrix}\;x,\,y\right]$$

where $|x| + |y| < 1$. (8)

For $b = c$, $p = q$ and $\alpha_1 = \beta_1\,,\,\ldots\,,\,\alpha_p = \beta_p$, (8) reduces to

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n\,(\mu)_n}{(\rho)_n\,n\,!}\,(1-x)^{n-a}\;{}_4F_3\left[\begin{matrix}1-\rho-n\,,\,-n\,,\,a\,,\,b'\,; \\ 1-\lambda-n\,,\,1-\mu-n\,,\,c'\,;\end{matrix}\;\frac{1}{(1-x)^2}\right]t^n$$

$$=(1-t)^{\lambda}\,F\left[\begin{matrix}\lambda:\rho-\mu\,;\,\mu\,; \\ \rho:\;\text{———}\;;\end{matrix}\;\frac{t}{t-1}\,,\,\frac{xt}{t-1}\right]F^{(2)}\left[\begin{matrix}a\,,\,\text{——}\,;\,b'\,; \\ \text{———}\,;\,c'\,;\end{matrix}\;x\,,\,t\right] \qquad (9)$$

Case IV

Put $$\Phi_j = \frac{(a)_j\,(b)_j}{(c)_j\,j\,!}\;{}_2F_1\left[\begin{matrix}a'\,,\,b'\,; \\ c+j\,;\end{matrix}\;x\right]$$

in (1) and (2) to get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda)_{n-k}\,(\mu)_{n-k}\,(a)_k\,(b)_k}{(\rho)_{n-k}\,(c)_k\,(n-k)\,!\,k\,!}\;{}_{p+1}F_q\left[\begin{matrix}-n+k\,,\,(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}a'\,,\,b'\,; \\ c+k\,;\end{matrix}\;x\right]t^n$$

$$=(1-t)^{-\lambda}\,F\left[\begin{matrix}\lambda:\rho-\mu\,;\,\mu\,,\,(\alpha_p)\,; \\ \rho:\;\text{———}\;;\;(\beta_q)\,;\end{matrix}\;\frac{t}{t-1},\,\frac{xt}{t-1}\right]F^{(3)}[a,\,a'\,;\,b,\,b'\,;\,c\,;\,x,\,t]$$

where $|x| + |t| < 1$. (10)

Case V

Next, put $$\Phi_j = \left\{\frac{(a)_j\,(b)_j}{(c)_j\,j\,!}\right\}\;{}_2F_1\left[\begin{matrix}a+j,\,b+j\,; \\ c\,;\end{matrix}\;x\right]$$

in (1) and (2) to get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda)_{n-k}\,(\mu)_{n-k}\,(a)_k\,(b)_k}{(\rho)_{n-k}\,(c)_k\,(n-k)\,!\,k\,!}\;{}_{p+1}F_q\left[\begin{matrix}-n+k,\,(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\;x\right]\cdot{}_2F_1\left[\begin{matrix}a+k,\,b+k\,; \\ c\,;\end{matrix}\;x\right]t^n$$

$$=(1-t)^{-\lambda}\,F\left[\begin{matrix}\lambda:\rho-\mu\,;\,\mu,\,(a_p)\,; \\ \rho:\text{—}\,;\qquad(b_q)\,;\end{matrix}\;\frac{t}{t-1}\,,\,\frac{xt}{t-1}\right].\,F^{(4)}\,[a,\,b\,;\,c,\,c'\,;\,x,\,t] \qquad (11)$$

Case VI

Let

$$\Phi_j = \frac{(\alpha_1)_j\,(\beta_1)_j}{(\gamma_1)_j\,j\,!}\;F_3\,(\alpha_2,\,\alpha_3,\,\beta_2,\,\beta_1+j\,;\,\gamma\,;\,x,\,y)$$

and (2) then becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda)_{n-k}\,(\mu)_{n-k}\,(\alpha_1)_k\,(\beta_1)_k}{(\rho)_{n-k}\,(\gamma_1)_k\,(n-k)!\,k!}\;{}_{p+1}F_q\left[\begin{matrix}-n+k,\,(\alpha_p);\\ (\beta_q);\end{matrix}\,x\right]\cdot F_3\,[\alpha_2,\,\alpha_3;\,\beta_2,\,\beta_1+j;\,\gamma;\,x,y]\,t^n$$

$$=(1-t)^{-\lambda}\,F\left[\begin{matrix}\lambda:\,\rho-\mu\,;\,\mu,\,(\alpha_p);\\ \rho:-\,;\qquad\quad (\beta_q);\end{matrix}\,\frac{t}{t-1},\,\frac{xt}{t-1}\right]\cdot F_N(\alpha_1,\,\alpha_2,\,\alpha_3;\,\beta_1,\,\beta_2,\,\beta_1;\,\gamma_1,\gamma_2,\,\gamma_3;\,t,\,x,\,y) \quad (12)$$

Case VII

Let

$$\Phi_j=\frac{(\alpha_1)_j\,(\beta_1)_j}{(\gamma_1)_j\,j\,!}\cdot F_2\,(\alpha_1+j,\,\beta_1,\,\beta_2+j\,;\,\gamma_1,\,\gamma_3\,;\,x,\,y)$$

then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda)_{n-k}\,(\mu)_{n-k}\,(\alpha_1)_k\,(\beta_1)_k}{(\rho)_{n-k}\,(\gamma_1)_k\,(n-k)!\,k!}\;{}_{p+1}F_q\left[\begin{matrix}-n+k,\,(\alpha_p);\\ (\beta_q);\end{matrix}\,x\right]\cdot F_2\,(\alpha_1+j,\,\beta_1,\,\beta_2+j\,;\,\gamma_1,\,\gamma_3;\,x,\,y)t^n$$

$$=(1-t)^{-\lambda}\,F\left[\begin{matrix}\lambda:\,\rho-\mu\,;\,\mu,\,(\alpha_p);\\ \rho:-\,;\qquad\quad (\beta_q);\end{matrix}\,\frac{t}{t-1},\,\frac{xt}{t-1}\right]\cdot F_E(\alpha_1,\alpha_1,\alpha_1,\,\beta_1,\beta_2,\beta_2,\,\gamma_1,\gamma_2,\gamma_3;\,t,x,y) \quad (13)$$

Case VIII

Let

$$\Phi_j=\frac{(\alpha_1)_j\,(\beta_1)_j}{(\gamma_1)_j\,j\,!}\;F^{(4)}\,(\alpha_1+j,\,\beta_1\,;\,\gamma_1,\,\gamma_2+j\,;\,x,\,y)\,,$$

then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda)_{n-k}\,(\mu)_{n-k}\,(\alpha_1)_k\,(\beta_2)_k}{(\rho)_{n-k}\,(\gamma_2)_k\,(n-k)!\,k!}\;{}_{p+1}F_q\left[\begin{matrix}-n+k,\,(\alpha_p);\\ (\beta_q);\end{matrix}\,x\right]\cdot F^{(4)}\,(\alpha_1+k,\,\beta_1\,;\,\gamma_1,\,\gamma_2+k\,;\,x,\,y)t^n$$

$$=(1-t)^{-\lambda}\,F\left[\begin{matrix}\lambda:\,\rho-\mu\,;\,\mu,\,(\alpha_p);\\ \rho:-\,;\qquad\quad (\beta_q);\end{matrix}\,\frac{t}{t-1},\,\frac{xt}{t-1}\right]\cdot F_F\,(\alpha_1,\alpha_1,\alpha_1,\,\beta_1,\beta_2,\beta_1;\,\gamma_1,\gamma_2,\gamma_2;\,t,\,x,y). \quad (14)$$

The triple hypergeometric functions F_E, F_F and F_N have been defined by Saran[9]. The series F_E, F_F and F_N are absolutely convergent for $|t|<r$, $|x|<s$, $|y|<\lambda$ where the positive quantities r, s and λ are called associated radii of the convergence of the triple series $\Sigma\, A_{m,n,p}\, t^m\, x^n\, y^p$.

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