SOME SERIES FOR THE FOX's H-FUNCTION

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In this paper some finite series of Fox's H-function have been established and some recurrence relations for the H-function have been obtained. Since the finite series of Fox's H-function can be employed to obtain identities, recurrence relations and transformations of the H-function, therefore such series occupy a prominent place in the literature of generalized hypergeometric functions. Certain series of this type play an important role in the development of the theory of special functions.

On specialising the parameters the H-function can be reduced to Meijer's G-function, MacRobert's E-function and many other higher transcendental functions. Therefore, the results established in the paper are of general character and hence may encompass several cases of interest.

The H-function introduced by Fox[1], will be represented and defined as follows:

$$H_{p,\,q}^{m,\,n}\left[z\left|\begin{matrix}(a_1, e_1), \ldots\ldots\ldots, (a_p, e_p)\\(b_1, f_1), \ldots\ldots\ldots, (b_q, f_q)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js),\, z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)}\,ds, \tag{1}$$

where an empty product is interpreted as 1, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; e's and f's are all positive, and L is a suitable contour of Barnes type, such that the poles of $\Gamma(b_j-f_js)$ ($j=1,2,\ldots\ldots,m$) lie on the right-hand side of the contour and those of $\Gamma(1-a_j+e_js)$ ($j=1,2,\ldots\ldots,n$) lie on the left-hand side of the contour.

In what follows for sake of brevity

$$\sum_{j=1}^{p} e_j-\sum_{j=1}^{q} f_j\equiv A,\ \sum_{j=1}^{n} e_j-\sum_{j=n+1}^{p} e_j+\sum_{j=1}^{m} f_j-\sum_{j=m+1}^{q} f_j\equiv B,$$

and (a_p, e_p) represents the set of parameters $(a_1, e_1),\ldots\ldots,(a_p, e_p)$.

First Series—The first summation is

$$\sum_{r=0}^{u}\frac{(-u)_r\,(\beta-\alpha-\tfrac{1}{2}-u)_r}{r!\,(3/2-\beta+\alpha)_r}\,H_{p+2,\,q+2}^{m+1,\,n+1}\left[z\left|\begin{matrix}(1-\alpha+r-u,h),(a_p,e_p),(1-\alpha-r,h)\\(1-\beta-r+u,h),(b_q,f_q),(1-\beta+r,h)\end{matrix}\right.\right]$$

$$=2^{2u}\,(1+\alpha-\beta)_u\,/\,(2+2\alpha-2\beta)_u\,.$$

$$.\,H_{p+4,\,q+4}^{m+2,\,n+2}\left\{z\left|\begin{matrix}\left(1-\alpha-\frac{u}{2},h\right),\left(\frac{1}{2}-\alpha-\frac{u}{2},h\right),(a_p,e_p),(1-\alpha,h),\left(\frac{3}{2}-\beta,h\right);\\\left(1-\beta+\frac{u}{2},h\right),\left(\frac{3}{2}-\beta+\frac{u}{2},h\right),(b_q,f_q),(1-\beta,h),\left(\frac{1}{2}-\alpha,h\right)\end{matrix}\right.\right\}, \tag{2}$$

where h is a positive number and $A\leqslant 0$, $B>0$, $|\arg z|<B\pi/2$, $Re(\alpha)>0$, $Re(\beta)>0$, *Proof*—Substituting on the left from (1), we have

$$\sum_{r=0}^{u}\frac{(-u)_r\,(\beta-\alpha-1/2-u)_r}{r!\,(3/2-\beta+\alpha)_r}\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\Gamma(\alpha-r+u+hs)\Gamma(1-\beta-r+u-hs)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)\Gamma(1-\alpha-r-hs)\,\Gamma(\beta-r+hs)}\,.\,z^sds.$$

Interchanging the order of Integration and summation and using[2], the series becomes,

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\,\Gamma(\alpha+u+hs)\,\Gamma(1-\beta+u-hs)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)\,\Gamma(1-\alpha-hs)\Gamma(\beta+hs)}\,.$$

$$.\,{}_4F_3\left[\begin{matrix}-u,\alpha+hs,1-\beta-hs,\ \beta-\alpha-1/2-u\\ 1-\alpha-u-hs,\ \beta-u+hs,\ 3/2-\beta+\alpha\end{matrix};1\right].\,z^s\,ds.$$

Now using[3], viz.

$${}_4F_3\left[\begin{matrix}-m,a,b,1/2-a-b-m\\ 1-a-m,1-b-m,1/2+a+b\end{matrix};1\right]=\frac{(2a)_m\,(2b)_m\,(a+b)_m}{(a)_m\,(b)_m\,(2a+2b)_m},$$

the right-hand side of (2) is obtained.

In (2), taking $u=1$, we obtain the relation

$$H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(1-\alpha,h),(a_p,e_p),(-\alpha,h)\\(1-\beta,h),(b_q,f_q),(2-\beta,h)\end{matrix}\right.\right]=H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(-\alpha,h),(a_p,e_p),(1-\alpha,h)\\(2-\beta,h),(b_q,f_q),(1-\beta,h)\end{matrix}\right.\right],$$

which is the particular case of

$$H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(\alpha,h),(a_p,e_p),(\alpha\pm\sigma,h)\\(\beta,h),(b_q,f_q),(\beta\pm\sigma,h)\end{matrix}\right.\right]$$

$$=H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(\alpha\pm\sigma,h),(a_p,e_p),(\alpha,h)\\(\beta\pm\sigma,h),(b_q,f_q),(\beta,h)\end{matrix}\right.\right],\qquad(3)$$

which can be established by expressing the H-function as Mellin-Barnes type integral and adjusting Gamma-functions suitably.

In (2) putting $u=2$ and using (3), we have

$$H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(-\alpha-1,h),(a_p,e_p),(1-\alpha,h)\\(3-\beta,h),(b_q,f_q),(1-\beta,h)\end{matrix}\right.\right]+\frac{(5/2-\alpha+\beta)}{(3/2+\alpha-\beta)}\,.$$

$$.\,H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(-\alpha,h),(a_p,e_p),(-\alpha,h)\\(2-\beta,h),(b_q,f_q),(2-\beta,h)\end{matrix}\right.\right]=2\,(2+\alpha-\beta)\,|\,(3/2+\alpha-\beta)\,.$$

$$.\,H^{m+2,n+2}_{p+4,q+4}\left[z\left|\begin{matrix}(-\alpha,h),(-\alpha-1/2,h),(a_p,e_p),(1-\alpha,h),(3/2-\beta,h)\\(2-\beta,h),\ (5/2-\beta,h),(b_q,f_q),(1-\beta,h),\ (1/2-\alpha,h)\end{matrix}\right.\right],$$

which can be verified by expressing the H-function as Mellin-Barnes type integral and adjusting Gamma-functions suitably.

Note—In further series also the result obtained with $u=1$ and $u=2$ can similarly be verified by expressing H-functions as Mellin-Barnes type integral and adjusting the Gamma-functions suitably.

Second Series—The second summation is

$$\sum_{r=0}^{u}\frac{(-u)_r\,(\beta-\alpha-1/2-u)_r}{r!\,(1/2+\alpha-\beta)_r}\,.\,H^{m+1,n+1}_{p+2,q+2}\left[z\left|\begin{matrix}(1-\alpha-r,h),(a_p,e_p),(1-\alpha+r-u,h)\\(1-\beta-r+u,h),(b_q,f_q),(1-\beta+r,h)\end{matrix}\right.\right]$$

$$=\frac{2^{2u}\,(1+\alpha-\beta)_u}{(2\alpha-2\beta+1)_u}\qquad(4)$$

$$\cdot H^{m+2,\,n+3}_{p+5,\,q+5}\left[z\left|\begin{matrix}(1-\alpha,h),\left(1-\alpha-\frac{u}{2},h\right),\left(\frac{1}{2}-\alpha-\frac{u}{2},h\right),(a_p,e_p),(1-\alpha-u,h),\left(\frac{3}{2}-\beta,h\right)\\ \left(1-\beta+\frac{u}{2},h\right),\left(\frac{3}{2}-\beta+\frac{u}{2},h\right),(b_q,f_q),(1-\beta,h),\left(\frac{1}{2}-\alpha,h\right),(1-\alpha-u,h)\end{matrix}\right.\right],$$

where h is a positive number and $A \leqslant 0$, $B > 0$, $|\arg z| < B\pi/2$, $Re(\alpha) > 0$, $Re(\beta) > 0$.

Third Series—The third summation is

$$\sum_{r=0}^{u}\frac{(-u)_r\,(1/2-\alpha+\beta-u)_r}{r!\,(\alpha-\beta+1/2)_r}\cdot H^{m+1,\,n+1}_{p+2,\,q+2}\left[z\left|\begin{matrix}(2-\alpha+r-u,h),(a_p,e_p),(1-\alpha-r,h)\\(1-\beta+r,h),(b_q,f_q),(1-\beta-r+u,h)\end{matrix}\right.\right]$$

$$=2^{2u}(\alpha-\beta)_u\,/\,(2\alpha-2\beta)_u\,.$$

$$\cdot H^{m+3,\,n+3}_{p+6,\,q+6}\left[z\left|\begin{matrix}\left(1-\alpha-\frac{u}{2},h\right),\left(\frac{3}{2}-\alpha-\frac{u}{2},h\right),(2-\alpha,h),(a_p,e_q),(1-\alpha,h),(3/2-\beta,h),\\(1-\beta+u,h);\\(1-\beta,h),\left(1-\beta+\frac{u}{2},h\right),\left(\frac{3}{2}-\beta+\frac{u}{2},h\right),(b_q,f_q),(1-\alpha,h),(3/2-\alpha,h),\\(1-\beta+u,h)\end{matrix}\right.\right],\quad (5)$$

where h is a positive number and $A \leqslant 0$, $B > 0$, $|\arg z| < B\pi/2$, $Re(\alpha) > 0$, $Re(\beta) > 0$.

Fourth Series—The fourth summation is

$$\sum_{r=0}^{u}\frac{(-u)_r(1/2-\alpha+\beta-u)_r}{r!\,(3/2+\alpha-\beta)_r}\,H^{m+2,\,n}_{p+2,\,q+2}\left[z\left|\begin{matrix}(a_p,e_p),(1-\alpha+r-u,h),(1-\alpha-r,h)\\(1-\beta+r,h),(1-\beta+u-r,h),(b_q,f_q)\end{matrix}\right.\right]$$

$$=2^{2u}\frac{(1+\alpha-\beta)_u\,(\frac{1}{2}+\alpha-\beta)_u}{(3/2+\alpha-\beta)_u\,(1+2\alpha-2\beta)_u}\,.$$

$$\cdot H^{m+3,\,n+2}_{p+5,\,q+5}\left[z\left|\begin{matrix}\left(1-\alpha-\frac{u}{2},h\right),\left(\frac{1}{2}-\alpha-\frac{u}{2},h\right),(a_p,e_p),(1-\alpha,h),(1-\alpha-u,h),\left(\frac{3}{2}-\beta,h\right)\\(1-\beta,h),\left(1-\beta+\frac{u}{2},h\right),\left(\frac{3}{2}-\beta+\frac{u}{2},h\right),(b_q,f_q),\left(\frac{1}{2}-\alpha,h\right),(1-\alpha-u,h)\end{matrix}\right.\right],\quad (6)$$

where h is a positive number and $A \leqslant 0$, $B > 0$, $|\arg z| < B\pi/2$, $Re(\alpha) > 0$, $Re(\beta) > 0$.

Fifth Series—The fifth series is

$$\sum_{r=0}^{u}\frac{(-u)_r\,(\beta-\alpha-1/2-u)_r}{r!\,(\alpha-\beta+3/2)_r}\cdot H^{m,\,n+2}_{p+2,\,q+2}\left[z\left|\begin{matrix}(1-\alpha+r-u,h),(1-r-\alpha,h),(a_p,e_p)\\(b_q,f_q),(1-\beta+r,h),(2-\beta-r+u,h)\end{matrix}\right.\right]$$

$$=\frac{2^{2u}(1+\alpha-\beta)_u}{(2+2\alpha-2\beta)_u}\cdot H^{m+2,\,n+3}_{p+5,\,q+5}\left[z\left|\begin{matrix}(1-\alpha,h),\left(1-\alpha\,\frac{u}{2},h\right),\left(\frac{1}{2}-\alpha-\frac{u}{2},h\right),(a_p,e_p),\\(2-\beta+u,h),(3/2-\beta,h);\\\left(\frac{3}{2}-\beta+\frac{u}{2},h\right),\left(2-\beta+\frac{u}{2},h\right),(b_q,f_q),(1-\beta,h),\\(2-\beta+u,h),(1/2-\alpha,h)\end{matrix}\right.\right],\quad (7)$$

where h is a positive number and $A \leqslant 0$, $B > 0$, $|\arg z| < B\pi/2$, $Re(\alpha) > 0$, $Re(\beta) > 0$.

Sixth Series—The sixth summation is

$$\sum_{r=0}^{u}\frac{(-u)_r}{r!}H^{m,\,n+3}_{p+3,\,q+3}\left[z\left|\begin{matrix}(-\alpha/2-r,h/2),\left(\frac{1-\alpha}{2}-r,\frac{h}{2}\right),(1-\beta-u-r,h),(a_p,e_p)\\(b_q,f_q),\left(-\beta/2-r,\frac{h}{2}\right),\left(\frac{1-\beta}{2}-r,\frac{h}{2}\right),(-\alpha-r,h)\end{matrix}\right.;\right]$$

$$=(\beta-\alpha)_u\,H^{m,\,n+4}_{p+4,q+4}\left[z\left|\begin{matrix}(-\alpha/2,h/2),\left(\frac{1-\alpha}{2},h/2\right),(-\beta,h)(1-\beta-24,h),(a_p,e_p);\\(b_q,f_q),(-\beta/2,h/2),\left(\frac{1-\beta}{2},h/2\right),(-\alpha,h),(-\beta-2u,h)\end{matrix}\right.\right], \qquad (8)$$

where h is a positive number and $A \leqslant 0$, $B > 0$, $|\arg z| < B\pi/2$, $Re(\alpha) > 0$, $Re(\beta) > 0$.

In closing, we note that all the series are finite and convergent, the conditions given only ensure the meaning of the H-functions. It is also interesting to note that on putting $u=2, 3, 4$, in all the series a large number of recurrence relations for the H-function may be obtained.

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