NUMERICAL COMPUTATION OF FLOW AND HEAT TRANSFER FROM AN ENCLOSED ROTATING DISC WITH SUCTION AND INJECTION

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(*Received* 20 *July*, 1978)

The Newton Raphson technique has been employed to solve the set of non-linear equations governing the problem of flow and heat transfer from an enclosed rotating disc. The disc called rotor is subjected to uniform injection while the top of the housing called stator, to an equal suction. The results for small Reynolds numbers are found in good agreement to that obtained earlier by series solution. The radial and transverse velocity profiles for large Reynolds numbers have been plotted in the regions of no recirculation. The effect of net radial inflow and outflow on temperature in the no-recirculation region has also been studied. The method is significant in this respect that it yields satisfactory results for large Reynolds numbers.

The phenomenon arising out of the flow and heat transfer over an enclosed rotating disc is encountered in many engineering applications viz. air cooling of turbine discs, pedestal bearing with central feeding of lubricant etc. The problem was first discussed by Soo[1] for viscous fluids. Later Sharma[2] suggested an improvement over Soo's formulation by considering the effects of circulation of the fluid (about the axis of rotation) induced by the presence of the shroud. The treatment with Non-Newtonian fluids was made by Sharma & Gupta[3] and Sharma & Sharma[4]. Soo[5] also studied the heat transfer part of the problem. A reappraisal of the heat transfer part was later made by Agarwal[6] and Sharma. Later Agarwal & Upmanyu[7] analysed the effects of uniform suction and injection on the flow and heat transfer by taking the modified velocity profile as suggested by Sharma[2]. However, in all these discussions the series solution valid for small Reynolds number have been considered. In the present paper we extend the solution for large Reynolds number using Newton Raphson technique. For the sake of comparison, the velocity and temperature functions for small Reynolds number have also been tabulated. Here it is worth mentioning that Newton Raphson method, being fast converging, is capable of solving quite difficult set of non-linear equations to a good degree of accuracy in few iterations.

FORMULATION OF THE PROBLEM

In a cylindrical set of reference, the system consists of a disc called rotor of radius r_1 rotating at a constant angular velocity in an incompressible viscous fluid, and is situated at a constant distance Z_0 ($< r_1$) from a stationary disc called stator, forming the part of a coaxial cylindrical casing or housing. The rotor is subjected to uniform injection W_0 and the stator to an equal suction. The symmetrical radial steady flow has a small mass rate of net radial outflow m ($m < 0$ for a net radial inflow). The inlet conditions are taken as the simple radial source flow along the z-axis, starting from radius r_0. The disc is kept at a constant temperature T_0 while that of the housing is T_1. The effect of injection and suction are governed by a non-dimensional parameter λ.

Let u, v, w be the radial, transverse and axial velocity components. The equations of continuity, momentum and energy for steady, axisymmetric flow are :

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u\right)+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right] \tag{2}$$

$$u \frac{\partial v}{\partial r} + \frac{\partial v}{\partial z} + \frac{uv}{r} = \nu \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right] \quad (3)$$

$$u \frac{\partial w}{r} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 w}{r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right] \quad (4)$$

$$\rho c_p \left[u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right] = K \left[\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial r^2} \right] + \phi \quad (5)$$

where ϕ, the viscous dissipation function is given by

$$\phi = 2 \rho \nu \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 + \frac{u^2}{r^2} + \frac{1}{2} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 \right]. \quad (6)$$

The boundary conditions are

for velocity :

$$u = 0,\ v = r\Omega,\ w = w_0 \text{ at } z = 0$$
$$u = v = 0,\ w = w_0 \text{ at } z = z_0 \quad (7)$$

($-w_0$ will denote the suction on the rotor and injection on the stator)

for temperature :

$$T = T_0 \text{ at } z = 0$$
$$T = T_1 \text{ at } z = z_0 \quad (8)$$

Introducing the non-dimensional variables

$$U = \frac{u}{\Omega z_0},\ V = \frac{v}{\Omega z_0},\ W = \frac{w}{\Omega z_0},\ P = \frac{P}{\Omega^2 z_0^2 \rho}$$
$$T^* = \frac{c_p}{\nu \Omega} T,\ X = \frac{r}{z_0},\ \xi = \frac{z}{z_0} \text{ and } \phi^* = \frac{\phi}{\mu \Omega^2} \quad (9)$$

the equations (2) — (6) transform as

$$U \frac{\partial U}{\partial X} + W \frac{\partial U}{\partial \xi} - \frac{V^2}{X} = - \frac{\partial P}{\partial X} + \frac{1}{Rz} \left[\frac{\partial^2 U}{X^2} + \frac{1}{X} \frac{\partial U}{\partial X} + \frac{\partial^2 U}{\partial \xi^3} - \frac{U}{X^2} \right] \quad (10)$$

$$U \frac{\partial V}{\partial X} + W \frac{\partial V}{\partial \xi} + \frac{UV}{X} = \frac{1}{Rz} \left[\frac{\partial^2 V}{\partial X^2} + \frac{1}{X} \frac{\partial V}{\partial X} + \frac{\partial^2 V}{\partial \xi^2} - \frac{V}{X^2} \right] \quad (11)$$

$$U \frac{W}{\partial X} + W \frac{\partial W}{\partial \xi} = - \frac{\partial P}{\partial \xi} + \frac{1}{Rz} \left[\frac{\partial^2 W}{\partial X^2} + \frac{1}{X} \frac{\partial W}{\partial X} + \frac{\partial^2 W}{\partial \xi^2} \right] \quad (12)$$

$$U \frac{UT^*}{\partial X} + W \frac{\partial T^*}{\partial \xi} = \frac{1}{Pr\ Rz} \left[\frac{{}^2 T^*}{\partial X^2} + \frac{1}{X} \frac{\partial T^*}{X} + \frac{\partial^2 T^*}{\partial \xi^2} \right] + \phi^* \quad (13)$$

where Pr $(= \mu c_p / K)$, Rz $(= \Omega z^2{}_0 / K)$ are respectively Prandtl number and Reynolds number.

(*i*) *Solution of equation of motion*

Following Agarwal & Upmanyu[7], we take

$$U = - X H'(\xi) + \frac{Rm}{Rz} \frac{M'(\xi)}{X}$$
$$V = X G(\xi) + \frac{Rl}{Rz} \frac{L(\xi)}{X},\ W = 2 H(\xi) \quad (14)$$

where H, G, L and M are dimensionless functions. Also $Rm\,(= m/2\pi\rho\Omega z_0)$ and $Rl\,(= l/2\pi\Omega z_0)$ are called as Reynolds number for net radial outflow and circulatory flow respectively (Rm is negative for net radial inflow). The number m, denoting the small mass rate of radial outflow, is given by

$$m = 2\pi \int_0^{z_0} r\,u\,dz,$$

and m is a constant associated with induced circulatory flow due to the presence of the shroud. The square of dimensionless radii, at which there is no-recirculation for the case of net radial outflow ($m > 0$) and net radial inflow ($m < 0$), respectively satisfy

$$\text{(a)} \qquad Rm > 0\,, \left(\frac{\partial U}{\partial \xi}\right)_{\xi=0} \geqslant 0\,, \quad \left(\frac{\partial U}{\partial \xi}\right)_{\xi=1} \leqslant 0$$

$$\text{(b)} \qquad Rm < 0\,, \left(\frac{\partial U}{\partial \xi}\right)_{\xi=0} \leqslant 0\,, \quad \left(\frac{\partial U}{\partial \xi}\right)_{\xi=1} \geqslant 0$$

The boundary conditions (6) can be rewritten as

$$\begin{aligned} &H(0) = H(1) = \lambda, \quad H'(0) = H'(1) = 0, \\ &G(0) = 1, \quad G(1) = 0, \quad L(0) = L(1) = 0, \\ &M(0) = 0, \quad M(1) = 1, \quad M'(0) = M'(1) = 0; \end{aligned} \tag{15}$$

where $\lambda\,(= w_0/2\,\Omega z_0)$ is a non-dimensional parameter governing the effects of injection and suction.

We take

$$P = P_0(\xi) + X^2 P_1(\xi) + P_2(\xi) \log X \tag{16}$$

Using (14) and (16) and neglecting squares and higher powers of (Rm/Rz), (Rl/Rz) assumed small, the equations (10)-(12), after simplification, finally reduce to

$$H^{iv} = 2Rz\,(HH''' + GG') \tag{17}$$

$$M^{iv} = 2\,Rz\,(H'\,M'' + H\,M''' - G'L - GL') \tag{18}$$

$$G'' = 2\,Rz\,(HG' - H'G) \tag{19}$$

$$L'' = 2\,Rz\,(HL' + M'G) \tag{20}$$

where $L = (Rl/Rm)\,L$.

As a first step to the solution, the equations (17)-(20) are converted into corresponding finite difference equations by employing following finite difference scheme

$$\phi'_j = \frac{\phi_{j+1} - \phi_{j-1}}{2h} \tag{21a}$$

$$\phi''_j = \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} \tag{21b}$$

$$\phi'''_j = \frac{(-\phi_{j-2} + 2\phi_{j-1} - 2\phi_{j+1} + \phi_{j+2})}{2h^3} \tag{21c}$$

$$\phi_j^{iv} = \frac{(\phi_{j-2} - 4\phi_{j-1} + 6\phi_j - 4\phi_{j+1} + \phi_{j+2}}{h^4} \tag{21d}$$

where ϕ stands for any of the functions G, H, L and M. Thus the finite difference equations corresponding to (17)-(20) are

$$\begin{aligned} k_{1,j} = (H_{j-2} - 4H_{j-1} + 6H_j - 4H_{j+1} + H_{j+2}) - RzH_j\,h(-H_{j-2} + 2H_{j-1} - 2H_{j+1} + H_{j+2}) - \\ - RzG_j h^3\,(G_{j+1} - G_{j-1}) = 0 \end{aligned} \tag{22}$$

$$K_{2,j} = (M_{j-2} - 4M_{j-1} + 6M_j - 4M_{j+1} + M_{j+2}) - RzH_j h(-M_{j-2} + 2M_{j-1} - 2M_{j+1} + M_{j+2})$$
$$- 2Rz\, h\,(H_{j+1} - H_{j-1})\,(M_{j+1} - 2M_j + M_{j+1}) + (Rzh^3 L_j\,(G_{j+1} - G_{j-1}) +$$
$$+ Rzh^3\, G_j\,(L_{j+1} - L_{j-1}) = 0 \qquad (23)$$

$$K_{3,j} = (G_{j+1} - 2G_j + G_{j-1}) - RzH_j h(G_{j+1} - G_{j-1}) + RzG_j h\,(H_{j+1} - H_{j-1}) = 0 \qquad (24)$$

$$HK_{4,j} = (L_{j+1} - 2L_j + L_{j-1}) - RzH_j h\,(L_{j+1} - L_{j-1}) - Rz\, G_j\, h\,(M_{j+1} - M_{j-1}) = 0 \qquad (25)$$

where $3 \leqslant j \leqslant 21$.

The boundary conditions (15) can be written as

$$\begin{aligned} &H_2 = H_{22} = \lambda\,,\ H_{23} = H_{21}\,,\ H_1 = H_3 \\ &G_2 = 1\,,\ G_{22} = 0,\ L_2 = L_{22} = 0 \\ &M_2 = 0,\ M_{22} = 1,\ M_1 = M_3;\ M_{21} = M_{23} \end{aligned} \qquad (26)$$

The present method consists of calculating numerically the values G_j, H_j, L_j and M_j at the points $\xi = \xi_j$. The interval [0, 1] is divided into twenty equal parts with $h = 0.05$. The implicit scheme used here with central difference formulae yields good accuracy. The system given by (22)-(25) represents 76 non-linear equations. We consider 76 functions $K_{1,j}$, $K_{2,j}$, $K_{3,j}$ and $K_{4,j}$ as functions of 76 variables G_3, G_4 .... G_{21}, H_3 .... H_{21}, M_3............ M_{21}, L_3............ L_{21}. To start the iterative procedure, the first approximations to G_j, H_j, L_j and M_j is prescribed. Let the exact solution be $\bar{G}_3$ .... $\bar{G}_{21}$, $\bar{H}_3$ .... $\bar{H}_{21}$, $\bar{L}_3$.......... $\bar{L}_{21}$, $\bar{M}_3$ .... $\bar{M}_{21}$, then the differences $\triangle G_j$, $\triangle H_j$, $\triangle L_j$ and $\triangle M_j$ are calculated from

$$\bar{\phi_j} = \phi_j + \triangle\phi_j\,,\ 3 \leqslant j \leqslant 21 \qquad (27)$$

where ϕ can be G, H, L or M.

Expanding $K_{i,j}$, $i=1,\ldots\ldots..4$ as Taylor series and neglecting partial derivatives of order greater than one, the differences approximated by the following linear system

$$D \begin{bmatrix} \triangle G_3 \\ \cdot \\ \cdot \\ \cdot \\ \triangle G_{21} \\ \triangle H_3 \\ \cdot \\ \cdot \\ \cdot \\ \triangle H_{21} \\ \triangle L_3 \\ \cdot \\ \cdot \\ \cdot \\ \triangle L_{21} \\ \triangle M_3 \\ \cdot \\ \cdot \\ \cdot \\ \triangle M_{21} \end{bmatrix} + \begin{bmatrix} K_{1,3} \\ \cdot \\ \cdot \\ \cdot \\ K_{1,21} \\ K_{2,3} \\ \cdot \\ \cdot \\ \cdot \\ K_{2,21} \\ K_{3,3} \\ \cdot \\ \cdot \\ \cdot \\ K_{3,21} \\ K_{4,3} \\ \cdot \\ \cdot \\ \cdot \\ K_{4,21} \end{bmatrix} = 0 \qquad (28)$$

where D is the Jacobian matrix of order 76. A better approximation G, H, L and M is thus obtained. Th procedure is repeated till the desired accuracy is achieved. The value of H' and M' are also calculated numerically and finally U, V and W from (14) are computed. The skin friction on the rotor and the stator is given by

$$S_{f0} \cong \left(\frac{\partial U}{\partial \xi}\right)_{\xi=0} \tag{29a}$$

and

$$S_{f1} \cong \left(\frac{U}{\partial \xi}\right)_{\xi=1} \tag{29b}$$

(ii) Solution of the energy equation

Substituting (15) into the energy equation (13), we have

$$\left[\left(-XH' + \frac{Rm}{Rz}\frac{M'}{X}\right)\frac{\partial T^*}{\partial X} + 2H\frac{\partial T^*}{\partial \xi}\right] = \frac{1}{PrRz}\left[\frac{\partial^2 T^*}{\partial X^2} + \frac{1}{X}\frac{\partial T^*}{\partial X} + \frac{\partial^2 T^*}{\partial \xi^2}\right] +$$

$$+2\left[6H^{-12} + \frac{X^2}{2}(G'^2 + H''^2) + \frac{Rm}{Rz}(G' L' - H'' M'')\right] \tag{30}$$

where

$$L = \frac{Rt}{Rm} L$$

Introducing the dimensionless temperature

$$T^* = T_0^* + \phi(\xi) + X^2 \ (\xi) \tag{31}$$

and simplifying (30), we get

$$2\,Rz\,(H\phi' - 6H'^2) + 2\,Rm\,(M'\psi - G'\,L' + H''\,M'') = \frac{1}{Pr}(4\psi + \phi'') \tag{32}$$

$$Rz\,(-2H'\,\psi + 2H\psi' - G'^2 - H''^2) = \frac{1}{Pr}\psi'' \tag{33}$$

The boundary conditions (8) reduce to

$$\begin{aligned} &\text{at } \xi = 0, \phi(\xi) = 0, \psi(\xi) = 0 \\ &\text{at } \xi = 1, \phi(\xi) = \varpi, \psi(\xi) = 0 \end{aligned} \tag{34}$$

The finite difference equations corresponding to (32) and (33) are

$$\frac{1}{Pr}\left(\frac{\psi_{j+1} - 2\,\psi_j + \psi_{j-1}}{h^2} - 2H_j\frac{\psi_{j+1} - \psi_{j-1}}{2h}Rz + 2Rz\,H_j'\,\psi_j + \right.$$

$$+ Rz\,(G_j'^2 + H_j''^2) = 0 \tag{35}$$

$$(\phi_{j+1} - 2\phi_j + \phi_{j-1})/h^2\,Pr - 2Rz\,H_j\,(\phi_{j+1} - \phi_{j-1})/2h + \frac{4}{Pr}\psi_j +$$

$$+ 12\,Rz\,H_j^2 - 2\,Rm\,(\psi_j M_j' - G_j'\,L_j' + H_j''\,M_j'' = 0 \tag{36}$$

which are then solved again by the method explained earlier.

The flux from the rotor and stator are respectively

$$q_0 = -\frac{K}{z_0}\left(\frac{\partial T^*}{\partial \xi}\right)_{\xi=0} \tag{37a}$$

and

$$q_1 = \frac{K}{z_0}\left(\frac{\partial T^*}{\partial \xi}\right)_{\xi=1} \tag{37b}$$

NUMERICAL RESULTS

The numerical calculations have been made for $Rz = 0.4, 40$; $Rm = 0.02, -0.02$, $\lambda = 1, -1$; and $\bar{\omega} = 1$. The values of velocity components

$$U_{X_1}^{(+)}, \quad U_{X_1}^{(-)}, \quad V_{X_1}^{(+)}, \quad V_{X_1}^{(-)} \quad \text{given by}$$

$$U_{X_1}^{(+)} = \left[U\sqrt{\frac{Rz}{Rm}}\right] \text{at } X_1, \; U_{X_1}^{(-)} = \left[U\sqrt{\frac{Rz}{Rn}}\right] \text{at } X_1, \; V_{X_1}^{(+)} = \left[V\sqrt{\frac{Rz}{Rm}}\right] \text{at } X_1$$

$$V_{X_1}^{(-)} = \left[V\sqrt{\frac{Rz}{Rn}}\right] \text{at } X_1 \,; \; Rn = -Rm$$

and temperature for $Rz = 0{\cdot}04$, so obtained have been compared with the results obtained earlier by Agarwal & Upmanyu which are found in close agreement. For $Rz = 40$, the radial and transverse velocity profiles at maximum radii for no-recirculation for $Rm = 0.02$ and -0.02 are plotted in Fig. (1) – (4). It is observed that the plane of maxima for radial velocity is pulled near the rotor for $Rm > 0$ while it shifts towards the stator for $Rm < 0$. The transverse velocity decreases throughout the gap for all values of Rm. The skin friction on the rotor and the stator has been calculated.

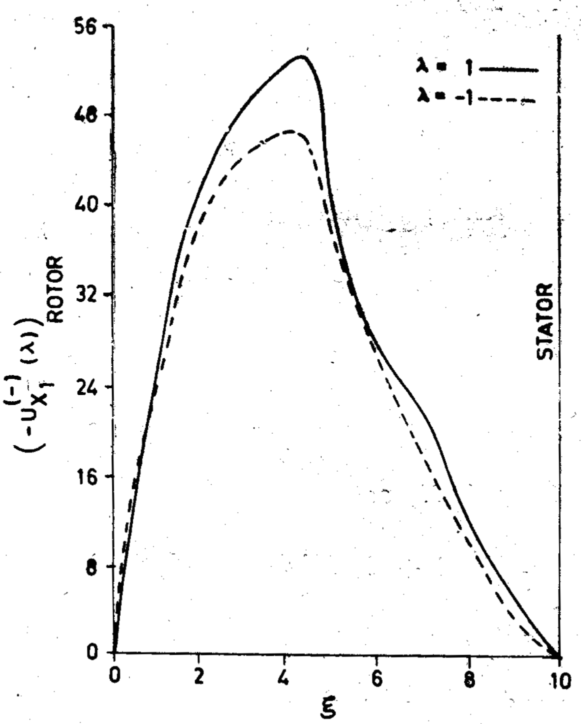

Fig. 1—Variation of radial velocity for maximum radii for no-recirculation (for $Rm > 0$).

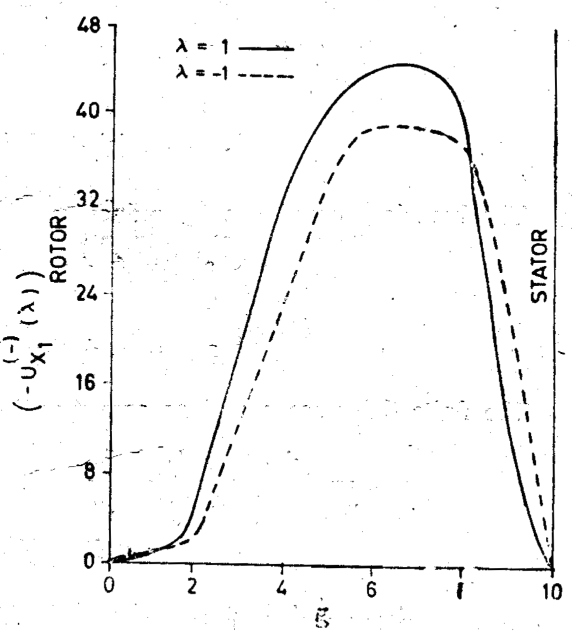

Fig. 2—Variation of radial velocity for maximum radii (for $Rm < 0$) for no-recirculation.

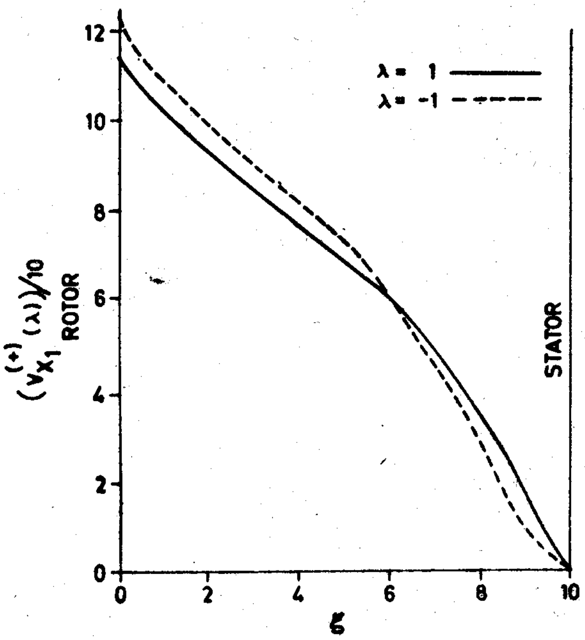

Fig. 3—Variation of transverse velocity for maximum radii (for $Rm > 0$) for no-recirculation.

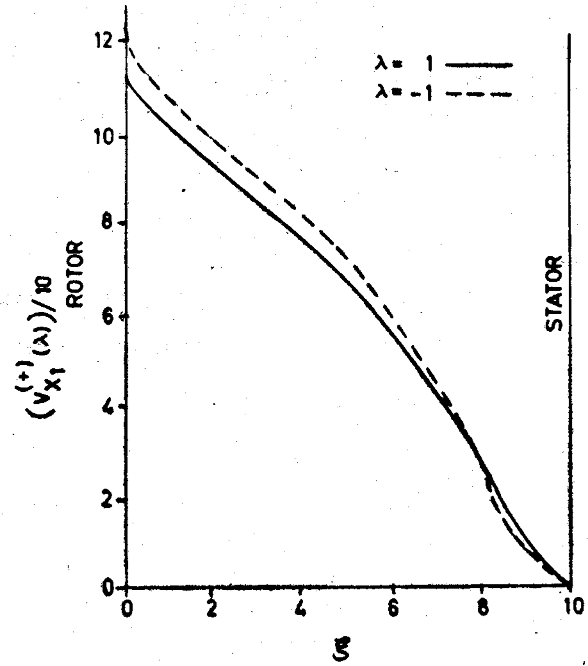

Fig.4—Variation of transverse velocity for maximum radii (for $Rm < 0$) for no-recirculation.

The temperature for $Pr = 20, Rz = 40$ and $X = 1.0$, is calculated and the results are shown in Fig. 5. It can be concluded that in no-recirculation region, there is heating near the rotor which is more pronounced for $Rm > 0$ and $\lambda = 1$; also cooling near the stator is increased for a negative Rm for the same λ. The heat flux on the rotor and the stator has also been found out.

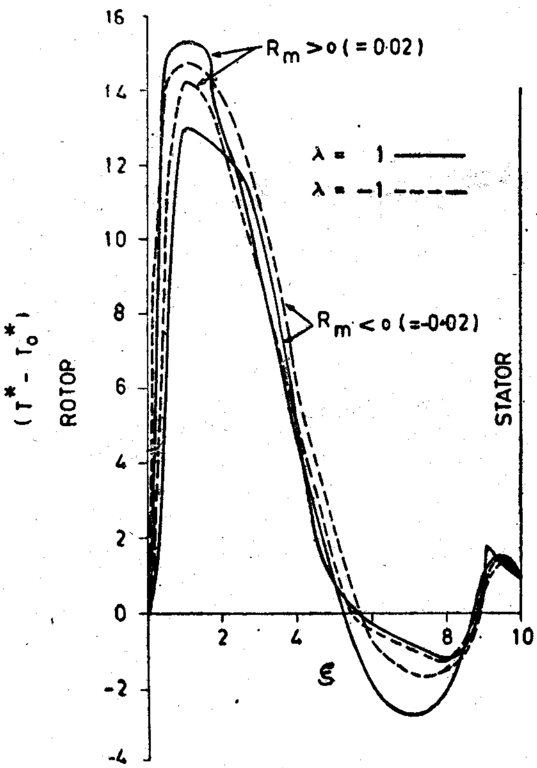

Fig. 5—Variation of $(T^*—T_0^*)$ for no-recirculation for $Rz = 40$, $x = 1·0$.

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