HEAT TRANSFER BETWEEN WELLSTIRRED AND QUIESCENT FLUIDS THROUGH A THICK VERTICAL PLATE

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In this paper an analytic solution has been obtained for a non-steady free convection on a thick vertical plate, resulting from heat transfer from a wellstirred fluid at the other face of the plate. Effects of the diffusivity ratio parameter k, and Prandtl number σ on heat flux at interface have been exhibited graphically.

NOMENCLATURE

l	=	thickness of the plate
T	=	absolute temperature
t	=	time
u	=	x component of velocity
ν	=	kinematic viscosity of quiescent fluid
x	=	vertical distance along the plate
y	=	horizontal distance from the vertical plate

Non-dimensional quantities

θ	=	$\frac{g\beta l^3 T}{\nu^2}$, β = coefficient of expansion
K	=	ratio of conductivities $\frac{K_2}{K_1}$
k	=	ratio of diffusivities $\frac{k_2}{k_1}$
K^*	=	$\sqrt{k}/K$
τ	=	$\nu t/l^2$
U	=	$\frac{ul}{\nu}$
Y	=	y/l, $Y_n = (\sqrt{k}\,Y + 2n + 1)$
α	=	$\frac{K_2}{K_3} \times \frac{k_3}{k_1} \times \frac{1}{\sigma}$
σ	=	Prandtl number ν/k

Subscripts

0	=	for initial conditions
1	=	for quiescent fluid
2	=	for solid
3	=	for well stirred fluid

Illingworth[1] carried out the study of non-steady free convection in a 'parallel type' flow to find short time solution after the onset of transition. Later Michiyoshi[2] and Siegel[3] studied the 'initial conduction regime' taking the temperature and velocity to be independent of distance along the vertical plate. Schetz & Eichhorn[4] and Manold & Young[5] simultaneously studied some additional cases of sinusoidal heat flux.

The author[6] studied the conjugate aspect of the problem for small values of time taking into account the thickness of the plate.

The present paper deals with the transient free convection in the quiescent fluid, resulting from the heat transfer from the well stirred fluid in contact with the other face of the thick plate. Following Carslaw & Jaeger[7], the face in contact with the well stirred fluid is assumed to remain at a constant temperature throughout. The problem has its application in calorimetry and heat exchangers.

PROBLEM

Taking x-axis along the vertical doubly infinite plate of thickness l, such that $-l \leqslant y \leqslant 0$, the well stirred and the quiescent fluid respectively occupy the space $y \leqslant -l$ and $y \geqslant 0$. Also $-\infty \leqslant \leqslant x \infty$ for both the solid and the fluid regions. Initial temperature of the solid and the quiescent fluid is zero and that of well stirred fluid T_0, which for $t > 0$ equals to the temperature of the surface of contact. For the problem stated the equations in the non-dimensional form can be written as

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial Y^2} + \theta_1 \tag{1}$$

$$\frac{\partial \theta_1}{\partial \tau} = \frac{1}{\sigma} \frac{\partial^2 \theta_1}{\partial Y^2} \tag{2}$$

and

$$\frac{\partial \theta_2}{\partial \tau} = \frac{k}{\sigma} \frac{\partial^2 \theta_2}{\partial Y^2} \tag{3}$$

with boundary conditions

$$\theta_1 = \theta_2 = U = 0 \; ; \; \theta_3 = \theta_0 \text{ for } \tau = 0 \qquad \text{(a)}$$

and for $\tau > 0$

$$\frac{\partial \theta_1}{\partial Y} = K \frac{\partial \theta_2}{\partial Y} \text{ at } Y = 0 \qquad \text{(b)}$$

$$\theta_1 = \theta_2 \text{ at } Y = 0 \qquad \text{(c)} \qquad (4)$$

$$\theta_2 = \theta_3 \text{ at } Y = -1 \qquad \text{(d)}$$

and

$$\frac{\partial \theta_3}{\partial \tau} = \frac{\alpha}{\sigma} \frac{\partial \theta_2}{\partial Y} \text{ at } Y = -1 \qquad \text{(e)}$$

Applying Laplace transform with respect to τ and putting the transformed quantities under the bar, these equations become

$$\frac{\partial^2 \bar{U}}{\partial Y^2} - p\bar{U} = -\bar{\theta}_1 \tag{5}$$

$$\frac{\partial^2 \bar{\theta}_1}{\partial Y^2} - p\sigma\bar{\theta}_1 = 0 \tag{6}$$

and

$$\frac{\partial^2 \bar{\theta}_2}{\partial Y^2} - \frac{p\sigma}{k} \bar{\theta}_2 = 0 \tag{7}$$

with transformed boundary conditions

$$\left.\begin{array}{ll} \dfrac{\partial\bar{\theta}_1}{\partial Y} = K\dfrac{\partial\bar{\theta}_2}{\partial Y} \text{ at } Y = 0 & \text{(a)} \\ \bar{\theta}_1 = \bar{\theta}_2 \quad \text{at} \quad Y = 0 & \text{(b)} \\ \text{and} & \\ p\bar{\theta}_2 - \dfrac{\alpha}{\sigma}\dfrac{\partial\bar{\theta}_2}{\partial Y} = \theta_0 \text{ at } Y = -1 & \text{(c)} \end{array}\right\} \quad (8)$$

SOLUTION

Solutions of equations (6) and (7) are

$$\bar{\theta}_1 = Ae^{-q_1 Y}$$

and

$$\bar{\theta}_2 = B\cosh q_2 Y + C\sinh q_2 Y$$

where

$$q_1^2 = p\sigma \quad \text{and} \quad q_2^2 = p\sigma/k$$

Evaluating the constants A, B and C using the boundary conditions 8(a), 8(b) and 8(c), we get

$$\bar{\theta}_1 = \sigma\theta_0\, e^{-q_1 Y}/\triangle \tag{9}$$

and

$$\bar{\theta}_2 = \sigma\theta_0\,(\cosh q_2 Y + K^*\sinh q_2 Y)/\triangle \tag{10}$$

where

$$\triangle = p\sigma\,(\cosh q_2 + K^*\sinh q_2) + \alpha q_2\,(\sinh q_2 + K^*\cosh q_2)$$

To get the solution valid for small values of time, we examine expressions in (9) and (10) for large values of p.

Thus

$$\bar{\theta}_1 = \frac{2\theta_0}{1+K^*}\sum_{n=0}^{\infty}(-1)^n\, K_1^{*n}\, e^{-q_2 Y_n}\left\{\frac{1}{p} - \frac{\alpha q_2}{\sigma p^2}(2n+1)\right\} \tag{11}$$

and

$$\bar{\theta}_2 = \frac{\theta_0}{1+K^*}\sum_{n=0}^{\infty}(-1)^n\left[K_1^{*n}\, e^{-q_2(2n+1-Y)}\left\{\frac{1}{p} - \frac{\alpha q_2}{\sigma p^2}(2n+1)\right\} + K_1^{*n+1}\, e^{-q_2(2n+1+Y)}\left\{\frac{1}{p} - \frac{\alpha q_2}{\sigma p^2}(2n+1)\right\}\right] \tag{12}$$

where

$$K_1^* = (1-K^*)/(1+K^*)$$

Term by term inversion of (11) and (12) gives

$$\theta_1 = \frac{2\theta_0}{1+K^*}\sum_{n=0}^{\infty}(-1)^n\, K_1^{*n}\left[\left\{\operatorname{erf} c\left(\frac{Y_n\sqrt{\sigma}}{2\sqrt{k\tau}}\right)\right\} - \frac{2\,\tau^{\frac{1}{2}}\,\alpha\,(2n+1)}{\sqrt{k\sigma}}\left\{i\ \operatorname{erf} c\left(\frac{Y_n\sqrt{\sigma}}{2\sqrt{k\tau}}\right)\right\}\right] \tag{13}$$

and

$$\theta_2 = \frac{\theta_0}{1+K^*} \sum_{n=0}^{\infty} (-1)^n K_1^{*n} \left[\left\{ \operatorname{erf} c \; \frac{(2n+1-Y)\sqrt{\sigma}}{2\sqrt{k\tau}} + K_1^* \operatorname{erf} c \; \frac{(2n+1+Y)\sqrt{\sigma}}{2\sqrt{k\tau}} \right\} - \frac{2\alpha\tau^{\frac{1}{2}}(2n+1)}{\sqrt{k\sigma}} \left\{ i \operatorname{erf} c \; \frac{(2n+1-Y)\sqrt{\sigma}}{2k\tau} + \right. \right.$$

$$\left. \left. + K_1^* \, i \operatorname{erf} c \; \frac{(2n+1+Y)\sqrt{\sigma}}{2\sqrt{k\tau}} \right\} \right] \quad (14)$$

where

$$i \operatorname{erf} c\,(x) = \frac{1}{\sqrt{\pi}} \, e^{-x} - x \operatorname{erf} c\,(x)$$

Also

$$\left. \frac{\partial \theta_1}{\partial Y} \right|_{Y=0} = \frac{2\theta_0}{1+K^*} \sum_{n=0}^{\infty} (-1)^n K_1^{*n} \left[-\exp\left(\frac{-Y^2_{n0}\sigma}{4k\tau} \right) \sqrt{\frac{\tau}{\pi\tau}} + \right.$$

$$+ \frac{\alpha(2n+1)}{\sqrt{k\pi}} \left\{ \exp\left(-\frac{Y_{n0}}{2} \sqrt{\frac{\sigma}{k\tau}} \right) + \sqrt{\pi} \operatorname{erf} c \; \frac{Y_{n0}\sqrt{\sigma}}{2\sqrt{k\tau}} - \right.$$

$$\left. - \frac{Y_{n0}\sqrt{\sigma}}{\sqrt{k}} \exp\left(Y^2_{n0}\,\sigma/4k\tau \right) \quad (15)$$

where $Y_{n0} = 2n+1$, represents the flux at the interface $Y = 0$.

Substituting the value of $\bar{\theta}_1$ from (15) in (9), we get a linear differential equation in $\bar{U}$. Its solution, under the condition of no slip at interface $Y = 0$ gives

$$\bar{U} = \frac{2\,\theta_0}{(\sigma-1)(1+K^*)} \sum_{n=0}^{\infty} (-1)^n K_1^{*n} \left\{ e^{-Y_\sigma q_2} - e^{-Y_n q_2} \right\} \left\{ \frac{1}{p^2} - \frac{\alpha q_2}{\sigma p^3} \right\} \quad (16)$$

where

$$Y\sigma = \left(\sqrt{\frac{k}{\sigma}} \; Y + 2n + 1 \right)$$

The inversion of equation (16) gives the velocity distribution of the fluid occupying the region $Y > 0$

$$U = \frac{2\theta_0}{(\sigma-1)(1+K^*)} \sum_{n=0}^{\infty} (-1)^n K_1^{*n} \left[\left\{ R_{1,Y^\sigma} - R_{1,Y_n} \right\} - \frac{\alpha}{\sqrt{k\sigma}} \left\{ R_{2,Y^\sigma} - R_{2,Y_n} \right\} \right] \quad (17)$$

where

$$R_{1,x} = \left(\tau + \frac{x^2\sigma}{2k} \right) \operatorname{erf} c \left(\frac{x\sqrt{\sigma}}{2\sqrt{k\tau}} \right) - x \left(\frac{\tau\sigma}{\pi k} \right)^{\frac{1}{2}} \exp\left(-x^2\sigma/4k\tau \right)$$

and

$$R_{2,x} = (4\tau)^{3/2}\, i^3 \operatorname{erfc}\left(x\sqrt{\sigma/2}\, \sqrt{k\tau} \right)$$

$$U\Big|_{\sigma=1} = \frac{2\theta_0 Y}{1-K^*} \sum_{n=0}^{\infty} (-1)^n K_1^{*n} \left[\frac{\sqrt{\tau}}{2\sqrt{\pi}} \exp(-Y_n^2/4k\tau) - \frac{\tau}{\sqrt{k}} \cdot \left\{ \left(\frac{1}{2} + \frac{Y_n}{2\sqrt{k\tau}} \right) \operatorname{erfc}(Y_n/2\sqrt{k\tau}) - \frac{5}{3\sqrt{\pi}} \frac{Y_n}{2\sqrt{k\tau}} \exp(-Y_n^2/4k\tau) \right\} \right] \quad (18)$$

DISCUSSION

Fig 1 depicts the heat flux at the interface $Y = 0$ at different times for different values of k. There is an overall increase in flux with increase in k. Also the peak values are reached earlier for greater k. It means that the greater the diffusivity ratio the earlier the flux will attain its maximum value, which is evident. Fig 2 shows the flux for different values of σ for $\sqrt{k} = 5$. It is seen that maximum value

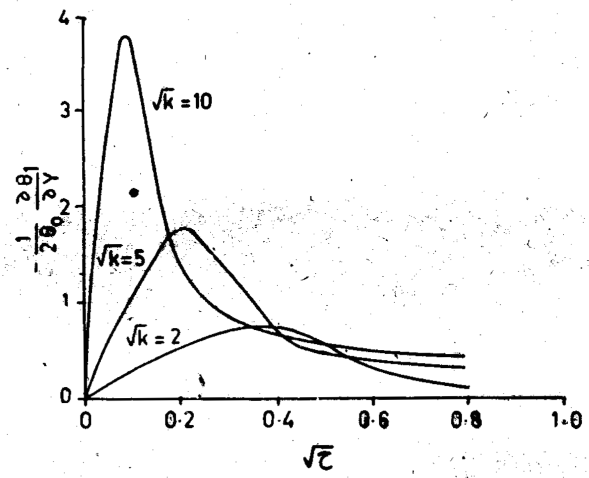

Fig 1—Flux at interface $Y=0$ at different timesfor various values $\sqrt{k}$, $\sigma=1$, $K^* = \cdot005$.

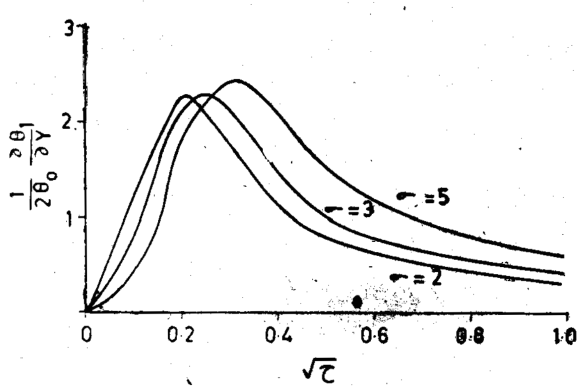

Fig 2—Flux at interface $Y=0$ at different times for various values of $\sqrt{\sigma}$, $\sqrt{k}=5$, $K^* = \cdot 005$.

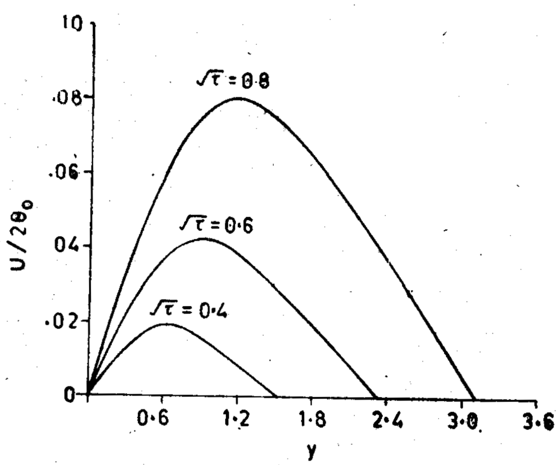

Fig 3—Velocity profiles for different values of $\sqrt{\tau}$, $\sqrt{k} = 10$, $\sigma=1$, $K^*=\cdot005$.

of flux is reached later for increased σ. Since $\sigma = \frac{\nu k}{k_2}$, so for a fixed k the increase in σ will mean either an increase in kinematic viscosity of the fluid or decrease in the diffusivity of the solid. Both will cause delayed diffusion of heat and hence delayed attainment of the maximum value of heat flux. Fig 3 exhibits the velocity profiles at different times for $\sigma = 1$. The profiles are seen to increase with the increase in time.

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