H-FUNCTION OF TWO VARIABLES VI

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Some integrals involving H-Function of two variables and G-Function of two variables, a special case of it, are evaluated. Integral formulae obtained contain various inetgrals as particular cases which seem new and quite general as the H-function of two variables includes most of the known functions of two variables as particular cases. The technique employed in evaluating the integrals is of Double Mellin transformation.

The H-Function of two variables, introduced by the author[1], includes most of the known functions in two variables as special cases. The G-Function of two variables[2], a particular case of it, has been frequently used in research publications by the author[4] and others. The G-Function of two variables not only includes Meijer's G-Function as particular case but also most of the known functions of two variables.

The object of this paper is to establish certain integral relations involving the H-Function and the G-Function of two variables.

NOTATIONS

The following notations will be used throughout the present series :

The symbol (ϵ_p, α_p) denotes the sequence of elements :

$$(\epsilon_1, \alpha_1), (\epsilon_2, \alpha_2), \ldots\ldots\ldots\ldots, (\epsilon_p, \alpha_p).$$

The H-Function of two variables[1] is expressed by the relation :

$$H^{n, \nu_1, \nu_2, m_1, m_2}_{p, [t:t'], s, [q:q']} \left[\begin{array}{c|l} x & (\epsilon_p, \alpha_p) \\ & (\gamma_t, r_t); (\gamma'_t, r'_t) \\ & (\delta_s, d_s) \\ y & (\beta_q, b_q); (\beta'_q, b'_q) \end{array} \right] = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Phi\{A(\xi+\eta)\}\, \psi(B\xi, C\eta) \cdot x \; y^{\eta} \, d\xi \, d\eta, \tag{1}$$

where

$$\Phi\{A(\xi+\eta)\} = \prod_{1}^{n} \Gamma(1-\epsilon_j + a_j\xi + a_j\eta) \{ \prod_{n+1}^{p} \Gamma(\epsilon_j - a_j\xi - a_j\eta) \prod_{1}^{n} \Gamma(\delta_j + d_j\xi + d_j\eta) \}^{-1},$$

$$\psi(B\xi, C\eta) = \prod_{1}^{m_1} \Gamma(\beta_j - b_j\xi) \prod_{1}^{\nu_1} \Gamma(\gamma_j + r_j\xi) \{ \prod_{m_1+1}^{q} \Gamma(1-\beta_j + b_j\xi) \prod_{\nu_1+1}^{t} \Gamma(1-\gamma_j - r_j\xi) \}^{-1} \cdot$$

$$\cdot \prod_{1}^{m_2} \Gamma(\beta'_j - b'_j\eta) \prod_{1}^{\nu_2} \Gamma(\gamma'_j + r'_j\eta) \{ \prod_{m_2+1}^{q'} \Gamma(1-\beta'_j + b'_j\eta) \prod_{\nu_2+1}^{t'} \Gamma(1-\gamma'_j - r'_j\eta) \}^{-1},$$

x, y are not equal to zero and empty product is interpreted as unity; $n, \nu_1, \nu_2, m_1, m_2, p, t, t', s, q, q'$ are integers sastisfying :

$$0 \leqslant n \leqslant p,\ 1 \leqslant m_1 \leqslant q,\ 1 \leqslant m_1 \leqslant q',\ 0 \leqslant \nu_1 \leqslant t,\ 0 \leqslant \nu_2 \leqslant t'; \tag{2}$$

$a_j\ (j=1, \ldots, p)$, $b_j\ (j=1, \ldots, q)$, $b'_j\ (j=1, \ldots, q')$, $r_j\ (j=1, \ldots, t)$, $r'_j\ (j=1, \ldots, t')$, $d_j\ (j=1, \ldots, s)$, are positive numbers and

$$\epsilon_j (j=1,\ldots,p),\ \beta_j (j=1,\ldots,q),\ \beta'_j (j=1,\ldots,q'),\ \gamma_j (j=1,\ldots,t),$$
$$\gamma'_j (j=1,\ldots,t'),\ \delta_j (j=1,\ldots,s),$$

are complex numbers such that

$$\left.\begin{array}{l} \alpha_i(\beta_h+\nu)\neq b_h(\epsilon_i-1-\lambda),\ (\lambda,\nu=0,1,\ldots;\ h=1,\ldots,m_1;\ i=1,\ldots,n), \\ \alpha_i(\beta_k'+\nu')\neq b_h'(\epsilon_i-1-\lambda'),\ (\lambda',\nu'=0,1,\ldots;\ h=1,\ldots m_2;\ i=1,\ldots,n). \end{array}\right\} \quad (3)$$

SOME KNOWN RESULTS

Some known results[1] which will be used in our present work, are given below :

$$H^{n,\nu,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c|l} x & (\epsilon_p,1) \\ & (\gamma_t,1);(\gamma'_{t'},1) \\ y & (\delta_s,1) \\ & (\beta_q,1);(\beta'_{q'},1) \end{array}\right] = G^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t);(\gamma'_{t'}) \\ y & (\delta_s) \\ & (\beta_q);(\beta'_{q'}) \end{array}\right], \quad (4)$$

where the function on the right side of (4) denotes the well-known G-Function of two variables[2].

$$H^{o,\nu_1,\nu_2,m_1,m_2}_{o,[t:t'],o,[q:q']}\left[\begin{array}{c|l} x & \cdots \\ & (\gamma_t,r_t);(\gamma'_{t'},r'_{t'}) \\ y & \cdots \\ & (\beta_q,b_q);(\beta'_{q'},b'_{q'}) \end{array}\right] = H^{m_1,\nu_1}_{t,q}\left[x\left|\begin{array}{l}(1-\gamma_t,r_t)\\(\beta_q,b_q)\end{array}\right.\right]\cdot$$
$$\cdot H^{m_2,\nu_2}_{t',q'}\left[y\left|\begin{array}{l}(1-\gamma'_{t'},r'_{t'})\\(\beta'_{q'},b'_{q'})\end{array}\right.\right], \quad (5)$$

where the functions on the right side of (5) denote the H-Functions due to Fox[3] and $n=0, p=0, s=0$.

$$H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c|l} x & (\epsilon_p,m) \\ & (\gamma_t,m);(\gamma'_{t'},m) \\ y & (\delta_s,m) \\ & (\beta_q,m);(\beta'_{q'},m) \end{array}\right] = \frac{1}{m^2}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c|l} x^{1/m} & (\epsilon_p) \\ & (\gamma_t);(\gamma'_{t'}) \\ y^{1/m} & (\delta_s) \\ & (\beta_q);(\beta'_{q'}) \end{array}\right], \quad (6)$$

where m is a positive integer and the right side of (6) is the G-Function defined by Agarwal[2].

EVALUATION OF INTEGRAL FORMULAE

We now establish the integral formulae involving the H-Function and the G-Function of two variables.

$$\int_0^\infty\int_0^\infty x^{\rho-1}y^{\rho'-1}\left\{H^{n,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c|l} zx^{\sigma} & (\epsilon_p,a_p) \\ & (\gamma_t,r_t);(\gamma'_{t'},r'_{t'}) \\ Z_1y^{\sigma_1} & (\delta_s,d_s) \\ & (\beta q,bq);(\beta'_{q'},b'_{q'}) \end{array}\right]\right\}\cdot$$
$$\cdot\left\{G^{n',\nu'_1,\nu'_2,m'_1,m'_2}_{p',[t_1:t'_1],s',[q_1:q'_1]}\left[\begin{array}{c|l} sx & (A'_p) \\ & (C_{t_1});(C'_{t'_1}) \\ s_1y & (D'_s) \\ & (B_{q_1});(B'_{q'_1}) \end{array}\right]\right\}dx\,dy$$

$$=(2\pi i)^{n'}\begin{pmatrix}\rho & \rho' \\ s & s_1\end{pmatrix}^{-\frac{1}{2}}\sum_{r=1}^{\infty}\left\{\left(e^{-i\pi(2r-1)\left[\sum_1^{n'}A_j+n'(\rho+\rho')\right]}\right)\cdot\right.$$

$$\cdot\left(H^{n,\nu_1+m'_1,\nu_2+m'_2,m_1+\nu'_1,m_2+\nu'_2}_{p+s',[t+q_1:t'+q'_1],s+p',[q+t_1:q'+t'_1]}\left[\begin{array}{c|l} zs^{-\sigma}e^{-i\pi\sigma(2r-1)} & \\ z_1s_1^{-\sigma_1}e^{-i\pi\sigma_1(2r-1)} & \end{array}\right.\right.$$

$$\left|\begin{matrix} (\epsilon_p, a_p), (D_s' - \rho - \rho', \sigma : \sigma_1) \\ (\gamma_{v_1}, r_{v_1}), (B_{q_1} + \rho, \sigma), (\gamma_{v_1+1,t}, r_{v_1+1,t}); (\gamma'_{v_2}, r'_{v_2}), (B'_{q_1'} + \rho', \sigma_1), (\gamma'_{v_2+1,t'}, r'_{v_2+1,t'}) \\ (\delta_s, d_s), (A_{p'} + \rho + \rho', \sigma : \sigma_1) \\ (\beta_{m_1}, b_{m_1}), (C_{t_1} - \rho, \sigma), (\beta_{m_1+1,q}, b_{m1+1,q}); (\beta'_{m_2}, b'_{m_2}), (C'_{t'_1} - \rho', \sigma_1), (\beta'_{m_2+1,q'}, b'_{m_2+1,q'}) \end{matrix}\right] \Bigg) \Bigg\} \quad (7)$$

provided that

$$R(\rho + B_i + \sigma\beta_h/b_h) > 0 \, (h = 1, \ldots, m_1; i = 1, \ldots, m_1');$$
$$R(\rho' + B_i' + \sigma_1 \beta'_h/b'_h) > 0 \, (h = 1, \ldots, m_2; i = 1, \ldots, m'_2);$$
$$R(\rho - C_j - \sigma\gamma_h/r_h) < 0 \, (j = 1, \ldots, v_1'; h = 1, \ldots, v_1); \sigma < 0;$$
$$R(\rho' - C'_j - \sigma_1 \gamma'_h/r'_h) < 0 \, (j = 1, \ldots, v'_2; h = 1, \ldots, v_2); \sigma_1 < 0;$$
$$p + q + s + t < 2(m_1 + v_1 + n), p + q' + s + t' < 2(m_2 + v_2 + n);$$
$$p' + q_1 + s' + t_1 < 2(m'_1 + v'_1 + n'), p' + q'_1 + s' + t'_1 < 2(m'_2 + v'_2 + n');$$
$$|\arg z| < \pi[m_1 + v_1 + n - \tfrac{1}{2}(p + q + s + t)],$$
$$|\arg z_1| < \pi[m_2 + v_2 + n - \tfrac{1}{2}(p + q' + s + t')];$$
$$|\arg s| < \pi[m'_1 + v'_1 + n' - \tfrac{1}{2}(p' + q_1 + s' + t_1)],$$
$$|\arg s_1| < \pi[m'_2 + v'_2 + n' - \tfrac{1}{2}(p' + q_1' + s' + t_1')].$$

Proof—

By substituting (1) for the H-Function of two variables in the left side of (7) and changing the order of integration (permissible by absolute convergence of the integrals involved), and using the exponential definition of cosec $\pi[(A_{n'}) + \rho + \rho' + \sigma\xi + \sigma_1\eta]$ and expanding it, we get, on term by term integration, the required result.

PARTICULAR CASES

Many more results can be derived by giving suitable values to the parameters in (7); some may even be new.

(*i*) Taking $(a_p) = 1, (r_t) = 1, (r'_{t'}) = 1, (d_s) = 1, (b_q) = 1, (b'_{q'}) = 1$,

and on using (4), we obtain

$$\int_0^\infty \int_0^\infty x^{\rho-1} y^{\rho'-1} G^{n, v_1, v_2, m_1, m_2}_{p, [t:t'], s, [q:q']} \left[\begin{matrix} zx^\sigma \\ z_1 y^{\sigma_1} \end{matrix} \middle| \begin{matrix} (\epsilon_p) \\ (\gamma_t); (\gamma'_{t'}) \\ (\delta_s) \\ (\beta_q); (\beta'_{q'}) \end{matrix}\right].$$

$$\cdot \, G^{n', v'_1, v'_2, m'_1, m'_2}_{p', [t_1:t'_1], s', [q_1:q'_1]} \left[\begin{matrix} sx \\ s_1 y \end{matrix} \middle| \begin{matrix} (A_{p'}) \\ (C_{t_1}); (C'_{t'_1}) \\ (D_s') \\ (B_{q_1}); (B'_{q'_1}) \end{matrix}\right] dx\, dy$$

$$= (2\pi i)^{n'} (s^{\rho} s_1^{\rho'})^{-1} \sum_{r=1}^{\infty} \left\{ \left(e^{-i\pi(2r-1)\left[\sum_1^{n'} A_j + n'(\rho + \rho')\right]} \right) \right. .$$

$$\cdot \left(H^{n, v_1 + m'_1, v_2 + m'_2, m_1 + v'_1, m_2 + v'_2}_{p + s', [t + q_1 : t' + q'_1], s + p', [q + t_1 : q' + t'_1]} \left[\begin{matrix} z s^{-\sigma} e^{-i\pi(2r-1)\sigma} \\ z_1 s_1^{-\sigma_1} e^{-i\pi(2r-1)\sigma_1} \end{matrix} \right| \right.$$

$$\left|\begin{matrix} (\epsilon_p), (D_s' - \rho - \rho', \sigma : \sigma_1) \\ (\gamma_{v_1}), (B_{q_1} + \rho, \sigma), (\gamma_{v_1+1,t}); (\gamma'_{v_2}), (B'_{q_1'} + \rho', \sigma_1), (\gamma'_{v_2+1,t'}) \\ (\delta_s), (A_p' + \rho + \rho^1, \sigma : \sigma_1) \\ (\beta_{m_1}), (C_{t_1} - \rho, \sigma), (\beta_{m_1+1,q}); (\beta'_{m_2}), (C'_{t'_1} - \rho', \sigma_1), (\beta'_{m_2+1,q'}) \end{matrix}\right] \Bigg) \Bigg\} . \quad (8)$$

(ii) Taking $n = 0, n' = 0, p = 0, p' = 0, s = 0, s' = 0,$ and using (5), we get

$$\int_0^\infty \int_0^\infty x^{\rho-1} y^{\rho'-1} H_{t,q}^{m_1,\nu_1}\left[zx^{\sigma} \middle| \begin{matrix} (1-\gamma_t, r_2) \\ (\beta_q, b_q) \end{matrix} \right] H_{t',q'}^{m_2,\nu_2}\left[z_1 y^{\sigma_1} \middle| \begin{matrix} (1-\gamma'_{t'}, r'_{t'}) \\ (\beta'_{q'}, b'_{q'}) \end{matrix} \right] \cdot$$

$$\cdot\, G_{t_1,q_1}^{m'_1,\nu'_1}\left[sx \middle| \begin{matrix} (1-C_{t_1}) \\ (B_{q_1}) \end{matrix} \right] G_{t'_1,q'_1}^{m'_2,\nu'_2}\left[s_1 y \middle| \begin{matrix} (1-C'_{t'_1}) \\ (B'_{q'_1}) \end{matrix} \right] dx\, dy$$

$$= (s^{\rho} s_1^{\rho'})^{-1} \sum_{r=1}^{\infty} \left\{ \left(e^{-i\pi(2r-1)} \right) \cdot \right.$$

$$\cdot \left(H_{t+q_1,\, q+t_1}^{m_1+\nu'_1,\, \nu_1+m'_1}\left[zs^{-\sigma} e^{-i\pi\sigma(2r-1)} \middle| \begin{matrix} (1-\gamma_{\nu_1}, r_{\nu_1}), (1-B_{q'_1}-\rho, \sigma), (1-\gamma_{\nu_1+1,t}, r_{\nu_1+1,t}) \\ (\beta_{m_1}, b_{m_1}), (C_{t_1}-\rho, \sigma), (\beta_{m_1+1,q}, b_{m_1+1,q}) \end{matrix} \right] \right) \cdot$$

$$\cdot \left. \left(H_{t'+q'_1,\, q'+t'_1}^{m_2+\nu'_2,\, \nu_2+m'_2}\left[z_1 s_1^{-\sigma_1} e^{-i\pi\sigma_1(2r-1)} \middle| \begin{matrix} (1-\gamma_{\nu_2}, r_{\nu_2}), (1-B'_{q'_1}-\rho', \sigma_1), (1-\gamma'_{\nu_2+1,t'}, r'_{\nu_2+1,t'}) \\ (\beta_{m_2}, b_{m_2}), (C'_{t'_1}-\rho', \sigma_1), (\beta'_{m_2+1,q'}, b'_{m_2+1,q'}) \end{matrix} \right] \right) \right\} \quad (9)$$

Similarly, we establish another integral formula analogous to (7), but it does not consist of an infinite series on the right side.

$$\int_0^\infty \int_0^\infty x^{\rho-1} y^{\rho'-1} H_{n'+p,[t:t'],s,[q:q']}^{n,\nu_1,\nu_2,m_1,m_2}\left[\begin{matrix} zx^{-\sigma} \\ z_1 y^{\sigma_1} \end{matrix} \middle| \begin{matrix} (\epsilon_p, a_p), (A_{n'}+\rho+\rho', \sigma:\sigma_1) \\ (\gamma_t, r_t); (\gamma'_{t'}, r'_{t'}) \\ (\delta_s, d_s) \\ (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{matrix} \right] \cdot$$

$$\cdot\, G_{p',[t_1:t_1'],s',[q_1:q_1']}^{n',\nu_1',\nu_2',m_1',m_2'}\left[\begin{matrix} sx \\ s_1 y \end{matrix} \middle| \begin{matrix} (A_{p'}') \\ (C_{t_1}); (C'_{t'_1}) \\ (D_s') \\ (B_{q_1}); (B'_{q'_1}) \end{matrix} \right] dx\, dy$$

$$= (s^{\rho} s_1^{\rho'})^{-1} \left\{ H_{p+s',[t+q_1:t'+q_1'],s+p',[q+t_1:q'+t_1']}^{n,\nu_1+m_1',\nu_2+m_2',m_1+\nu_1',m_2+\nu_2'} \left[\begin{matrix} z s^{-\sigma} \\ z_1 s_1^{-\sigma_1} \end{matrix} \right| \right.$$

$$\left. \left| \begin{matrix} (\epsilon_p, a_p), (D_{s'}-\rho-\rho', \sigma:\sigma_1) \\ (\gamma_{\nu_1}, r_{\nu_1}), (B_{q_1}+\rho, \sigma), (\gamma_{\nu_1+1,t}, r_{\nu_1+1,t}); (\gamma'_{\nu_2}, r'_{\nu_2}), (B'_{q_1'}+\rho', \sigma_1), (\gamma'_{\nu_2+1,t'}, r'_{\nu_2+1,t'}) \\ (\delta_s, d_s)(A_{n'+1,p'}+\rho+\rho', \sigma:\sigma_1) \\ (\beta_{m_1}, b_{m_1}), (C_{t_1}-\rho, \sigma), (\beta_{m_1+1,q}, b_{m_1+1,q}); (\beta'_{m_2}, b'_{m_2}), (C'_{t_1'}-\rho', \sigma_1), (\beta'_{m_2+1,q'}, b'_{m_2+1,q'}) \end{matrix} \right] \right\} \quad (10)$$

The conditions of validity for this result are same as in (7) as this is a direct take-off from that result.

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