ELASTIC DEFORMATION OF AN ORTHOTROPIC SEMI-INFINITE PLATE WITH STRAIGHT BOUNDARY ASYMMETRIC WITH RESPECT TO THE ELASTIC AXES OF THE MATERIAL UNDER UNIFORM PARTIAL LOADING

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The solution of the problem of an orthotropic semi-infinite plate with straight boundary asymmetric with respect to the elastic axes of the material has been obtained under the assumption that a part of the boundary near the origin is uniformly stressed and the rest is stress-free. The use of Beltrami Cayley's conformal mapping has been made. In particular, the solution has been obtained for the plate with a concentrated force.

The plane problems of plates of orthotropic materials have been treated so far by many investigators[1–3, 7–9]. These problems mainly concern with the domains symmetric with respect to the elastic axes of the material. In other cases, only few works have been reported upto the present by Lekhnitzkii[8, 9], Moriguchi[5], Washizu[6], Haguchi[4], Hayashi[10, 11] etc. This is mainly due to the difficulties encountered in the treatment of the differential equations in the case of the plates of orthotropic materials with symmetric elastic axes.

On the other hand, the stress functions for the orthotropic plate are usually given as the sum of two harmonic functions in two functional planes. These problems can be reduced to simpler problems by the use of proper conformal mapping. In order to find proper mapping functions, Beltrami Cayley's method can be used conveniently.

In this paper, problem has been solved by using Beltrami Cayley's conformal mapping. In particular, the solution for the plate with concentrated force has been deduced.

STATEMENT OF THE PROBLEM

We consider the small deformation of a semi-infinite plate of orthotropic material with boundary line inclined at an angle α to its elastic axis (x-axis). We take the $\xi - \eta$ coordinates as shown in Fig. 1. Then the x-y coordinates are expressed as

$$x = \xi \cos \alpha \, , \; y = \xi \sin \alpha \qquad (1)$$

It is assumed that a part of the boundary $(-a, a)$ is under the action of uniform pressure and the rest is stress-free. Therefore the boundary conditions on the real axis $\eta = 0$ are given as

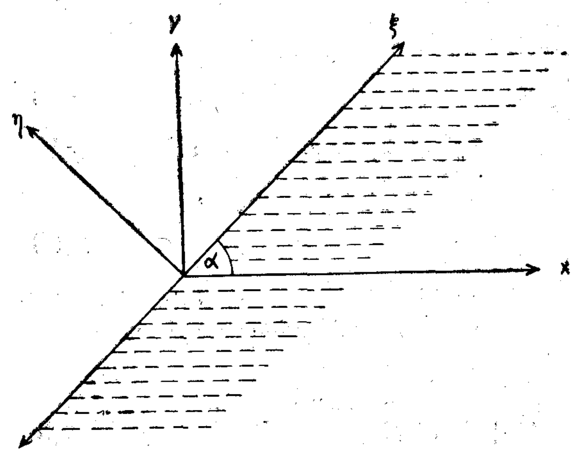

Fig. 1—x-y coordinates and ξ—η coordinates.

$$\left.\begin{array}{lll} \sigma_\eta = -P & & -a \leqslant \xi \leqslant a \\ \sigma_\eta = 0 & & |\xi| > a \\ \tau_{\xi\eta} = 0 & & -\infty < \xi < \infty \end{array}\right\} \qquad (2)$$

FUNDAMENTAL FORMULAE

Considering the orthogonal coordinates x-y in an orthotropic plate parallel to the elastic axes of the material, the two-dimensional stress components are given as[10]

$$\left.\begin{array}{l} \sigma x = -\text{Real}\,[p_1^2 f_1''(z_1) + p_2^2 f_2''(z_2)] \\ \sigma y = \text{Real}\,[f_1''(z_1) + f_2''(z_2)] \\ \tau xy = I_m\,[p_1 f_1''(z_1) + p_2 f_2''(z_2)] \\ z_j = x + i p_j y, \; j = 1, 2 \end{array}\right\} \qquad (3)$$

where f_j are the stress functions of z_j, dash denotes the differentiation with respect to z_j, and p_j are the real or complex conjugate constants concerning the orthotropic property of the material as shown below

$$p_1^2\, p_2^2 = E_x/E_y\,,\; p_1^2 + p_2^2 = (E_x/G_{xy}) - 2\,\nu_x$$

where E_x, E_y are the modulii of elasticity in x, y directions respectively, G_{xy} the modulus of rigidity, and ν_x the Poisson's ratio in x-direction.

The stress components in ξ—η directions inclined to the x—y axes at an angle α (Fig. 1) are given a

$$\begin{aligned}
\sigma_\xi &= -\,\text{Real}\left[\sum_{j=1,2}(p_j\cos\alpha + i\sin\alpha)^2 f_j''(z_j)\right]\\
&= -\,\text{Real}\left[\sum_{j=1,2}\frac{1}{4}\left\{(1+p_j)\,e^{i\alpha} - (1-p_j)\,e^{-i\alpha}\right\}^2 f_j''(z_j)\right]\\
\sigma_\eta &= \text{Real}\left[\sum_{j=1,2}(\cos\alpha + p_j\sin\alpha)^2 f_j''(z_j)\right]\\
&= \text{Real}\left[\sum_{j=1,2}\frac{1}{4}\left\{(1+p_j)\,e^{i\alpha} + (1-p_j)\,e^{-i\alpha}\right\}^2 f_j''(z_j)\right]\\
\tau_{\xi\eta} &= I_m\left[\sum_{j=1,2}(\cos\alpha + i\,p_j\sin\alpha)\,(p_j\cos\alpha + i\sin\alpha)\,f_j''(z_j)\right]\\
&= I_m\left[\sum_{j=1,2}\frac{1}{4}\left\{(1+p_j)^2\,e^{2i\alpha} - (1-p_j)^2\,e^{-2i\alpha}\right\} f_j''(z_j)\right]
\end{aligned} \tag{4}$$

where $\sum\limits_{j=1,2}$ means the summation for $j=1$ and 2.

SOLUTION OF THE PROBLEM

Since the boundary line on the physical plane is given by (1), z_j on the functional plane of the stress function take values

$$z_j = x + ip_j y = (\cos\alpha + ip_j\sin\alpha)\,\xi_j = \phi_j(\xi_j)\,. \tag{5}$$

Therefore the transformation function ϕ_j is defined as[10]

$$z_j = \phi_j(\zeta_j),\;\; \zeta_j = \xi_j + i\eta_j\,. \tag{6}$$

It is obvious from (5) and (6) that the real axis $\eta_j = 0$ in the ζ_j — plane corresponds to the boundary line in the physical plane and the following holds

$$\xi_1 = \xi_2 = t \tag{7}$$

Consequently, rewriting the stress functions in terms of ζ_j, the problem is transformed into a boundary value problem referring to the real axis on the ζ_j — plane. Hence the stress functions are given as

$$(\cos\alpha + ip_j\sin\alpha)\,f''_j(z_j) = \int_0^\infty \Big[\,a_j(t) + ib_j(t)\Big]\,e^{-i\zeta_j t}\,dt \tag{8}$$

where the unknown constants $a_j(t)$ and $b_j(t)$ are given as

$$a_j(t) = \frac{1}{\pi(p_j - p_k)} \int_{-\infty}^{\infty} \Big[\Big\{ \sigma_\eta(u) \sin\alpha - \tau_{\xi\eta}(u) \cos\alpha \Big\} \sin ut - $$

$$- p_k \Big\{ \sigma_\eta(u) \cos\alpha + \tau_{\xi\eta}(u) \sin\alpha \Big\} \cos ut \Big] du$$

$$b_j(t) = \frac{1}{\pi(p_j - p_k)} \int_{-\infty}^{\infty} \Big[\Big\{ \tau_{\xi\eta}(u) \cos\alpha - \sigma_\eta(u) \sin\alpha \Big\} \cos ut - $$

$$- p_k \Big\{ \sigma_\eta(u) \cos\alpha + \tau_{\xi\eta}(u) \sin\alpha \Big\} \sin ut \Big] du -$$

where $j = 1, 2$, $k = 1, 2$ $j \neq k$.

With the boundary condition (2), we find that

$$a_j(t) = \frac{2Pp_k \cos\alpha}{\pi(p_j - p_k)} \cdot \frac{\sin at}{t} \quad (9)$$

$$b_j(t) = \frac{2P \sin\alpha}{\pi(p_j - p_k)} \cdot \frac{\sin at}{t} \quad (10)$$

Hence the stress functions obtained by putting the values of $a_j(t)$ and $b_j(t)$ from (9) and (10) in (8) are a follows :

$$f_j''(z_j) = \frac{2P(p_k \cos\alpha + i \sin\alpha)}{\pi(p_j - p_k)(\cos\alpha + ip_j \sin\alpha)} \cdot \tan^{-1}\left(\frac{a}{i\xi_j}\right) \quad (11)$$

EXPRESSION FOR EDGE STRESSES

The stress components can be very easily evaluated with the help of the formulae (4) and the stress functions (11). However, on the edge of the plate for $\xi > a$, we have

$$\left.\begin{aligned} \sigma_\xi / P &= K \tanh^{-1}(a/\xi) \\ \sigma_\eta &= \tau_{\xi\eta} = 0 \end{aligned}\right\} \quad (12)$$

where $$K = \frac{2(p_1 + p_2)(p_1 p_2 - 1) \sin\alpha \cos\alpha}{\pi(\cos^2\alpha + p_1^2 \sin^2\alpha)(\cos^2\alpha + p_2^2 \sin^2\alpha)}$$

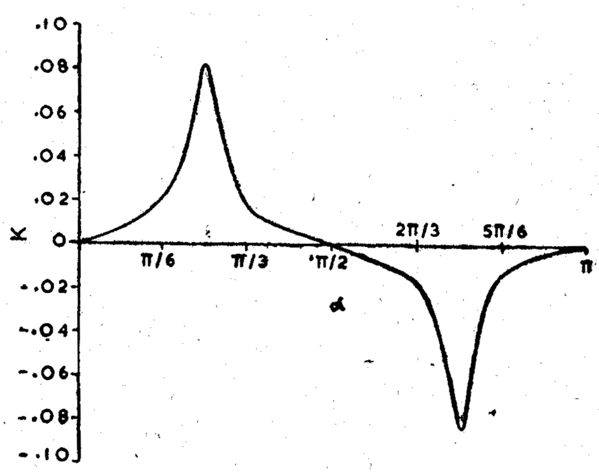

Fig. 2—K vs α for an Oak plate with $p_1 = 5/3$ and $p_2 = 49/51$.

The value of K remains constant for a particular plate but varies as α, the angle between the elastic axes of the material and the physical boundary line, takes different values. For an Oak plate with $p_1 = 5/3$, $p_2 = 49/51$, the value of K for different values of α has been given below and graphically represented in Fig. 2.

α	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
K	0·00	0·14	0·87	0·15	0·00	—0·15	—0·87	—0·14	0·00

DEDUCTION OF THE SOLUTION FOR THE PLATE UNDER THE ACTION OF NORMAL CONCENTRATED FORCE AT THE ORIGIN

In case, the plate is under the action of a normal concentrated F at the origin, the solution may be very easily deduced from above by taking the limit as $a \to 0$. While F is given as

$$\underset{a \to 0}{\text{Lt}} \; 2a\,P = F,$$

The stress functions (11) reduce to

$$f_j''(z_j) = \frac{F(p_k \cos\alpha + i \sin\alpha)}{\pi(p_j - p_k)(\cos\alpha + ip_j \sin\alpha)\, i\xi_j} \tag{13}$$

The function (13) is in agreement with that given by Hayashi[10]. The stress components (12) reduce to

$$\sigma_\xi / F = K / \xi \,; \; \sigma_\eta = \tau_{\xi\eta} = 0$$

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