ON COMPRESSIBLE MAGNETO-RELATIVISTIC FLOWS

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This paper investigates the character of compressible magneto-relativistic flow in subsonic, supersonic and transonic regions by applying the method of small amplitude disturbance solutions. Relations between the pressure coefficient, free stream Mach number and Alfven speed in these regions have also been obtained.

Harris[1] has investigated magneto-relativistic equations for a perfect fluid and has shown that for a conducting fluid in the presence of magnetic field there exist three wave velocities. Taub[2-5], has treated the macroscopic theory of relativistic fluid dynamics. In this paper we have investigated the character of compressible magneto-relativistic flow in subsonic, supersonic and transonic regions by writing the governing equations in a form which is suitable for finding small amplitude disturbance solutions in these three regions. Relations between the pressure coefficient, free stream Mach number and Alfven speed in these regions have also been obtained. It has been shown that in the absence of any magnetic and relativistic effects, these relations reduce to Glauert law [6], Ackeret solution[6] and ordinary transonic expression obtained by Spreiter & Alksne[7] respectively.

BASIC EQUATIONS AND THEIR ANALYSIS

The stress energy tensor T^{ij} of a perfect magneto-relativistic fluid neglecting heat conductivity and viscosity is given by[8,9]

$$T^i{}_j = \sigma\, u^i\, u_j - p\, g^i{}_j + \tau^i{}_j + (\epsilon\mu - 1)\, \tau^i{}_l\, u^l\, u_j \tag{1}$$

where

$$\tau^i{}_j = \tfrac{1}{4}\, \mu^{-1}\, g^i{}_j\, H^{mn}\, H_{mn} - \mu^{-1}\, H^{mi}\, H_{mj} \tag{2}$$

is the electromagnetic stress energy tensor,

$$\sigma = \rho\eta = \rho\,(1 + e + p/\rho)$$

where p is the pressure, ρ the density, μ the magnetic permeability, e the internal energy, ϵ the dielectric constant and H^{ij} is the electro-magnetic field tensor defined by $H^{ij} = \epsilon^{ijkl}\, U_k\, H_l$ where ϵ^{ijkl} is the usual permutation tensor. The maxwell's equations are then, $H_{ij,k} + H_{jk,i} + H_{ki,j} = 0$. The Latin affixes take values $1, 2, 3, 4$ whereas the Greek affixes 1, 2, 3. The covariant metric tensor $g_{ij} = 0$ for $i \neq j$ and $g_{11} = g_{22} = g_{33} = -g_{44} = -1$. In rectangular cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ and $x^4 = t$, where t is the time coordinate.

Also $u^i \overset{def}{=\!=} \dfrac{dx^i}{ds}$ is the four dimensional velocity vector of the fluid satisfying,

$$g_{ij}\, u^i\, u^j = u_i\, u^i = 1 \tag{3}$$

and the metric of the space time is given by

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{4}$$

Since for values of ϵ and μ in vacuum, the entropy of the fluid particle is constant along a streamline and $\epsilon\mu = 1$, equation (1) reduces to,

$$T^i{}_j = \sigma\, u^i\, u_j - p g^i{}_j + \tau^i{}_j \tag{5}$$

In rectangular cartesian coordinates x, y, z, the ordinary resultant velocity V is given by

$$V^2 = u^2 + v^2 + w^2 \tag{6}$$

If u_α be the ordinary 3-velocity components, the relations between 3-velocity and 4-velocity are,

$$u_1 = u\beta,\ u_2 = v\beta,\ u_3 = w\beta,\ u_4 = \beta \tag{7}$$

where $$\beta = 1/(1-V^2)^{\frac{1}{2}}$$

The equations governing the motion of the fluid are,

$$(\rho\, u^i)_{,i} = 0 \tag{8}$$

$$(T^{ij})_{,j} = 0 \tag{9}$$

$$u_i\, H^{ij} = 0 \tag{10}$$

where a *comma* preceding an index denotes partial derivative with respect to x_i. Equation (8) is the relativistic form of the conservation of mass, equation (9) is the magneto-relativistic generalization of the conservation of momentum and energy and (10) is the field equation. Substituting the value of T^{ij} from (5) into (9) we get,

$$\sigma\, u^j\, u^i{}_{,j} = (g^{ij} - u^i\, u^j)\, p_{,j} + \mu\, (H_k\, H^k\, u^i\, u^j + H^i\, H^j + \tfrac{1}{2}\, g^{ij}\, H_k\, H^k)_{,j} \tag{11}$$

In the equation (11) the three space components are the generalizations of Euler's equation for the magneto-relativistic case and the fourth equation gives the time component. Multiplying the fourth equation by the 3-velocity u_α and subtracting it from the other three we have,

$$\beta^2\, (\sigma - \mu\, H_\theta\, H^\theta)\ \frac{Du_\alpha}{Dt} = -\frac{\partial p}{\partial x_\alpha} - u_\alpha\, \frac{\partial p}{\partial t} + \frac{3}{2}\, \mu\, (H_\alpha\, H^\alpha),\, \beta \tag{12}$$

The equation (8) in terms of 3-velocity can be written as,

$$\beta\, \frac{D\rho}{Dt} + \rho \left[\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x_\alpha}\, (\beta\ u_\alpha) \right] = 0 \tag{13}$$

where $\dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + u_\alpha\, \dfrac{\partial}{\partial x_\alpha}$ denotes the differentiation following the fluid. Equations (12) and (13) are the magneto-relativistic flow equations in terms of 3-velocity. For characterizing the fluid, the equation of state is taken as

$$e = e\,(p,\, \rho) \tag{14}$$

It has been shown by Taub[3] that for gases in special relativity, the internal energy e is given by

$$1 + e \geqslant 3\, p/2\, \rho + \left[\, 1 + 9\, p^2/4\ \ \rho^2 \,\right]^{\frac{1}{2}} \tag{15}$$

where for the equality sign in (15) the gas is said to be limiting and when $p/\rho \gg 1$, the gas is said to be hot. The gas is known as degenerate if it is both limiting and hot. Thus in this case, the internal energy of the gas is given by

$$1 + e = 3\, p/\rho \tag{16}$$

The internal energy defined by

$$e = p/\rho\ \ (\gamma - 1) \tag{17}$$

with equation (16) give $\gamma = 4/3$ for a degenerate gas, whereas equation (17) with $\gamma \geqslant 5/3$ is not permitted for a relativistic gas. Since for small amplitude disturbance flow, γ is a slowly varying function of p/ρ therefore in this paper we take γ to be a constant for simplicity in the analysis. In the case of relativistic isentropic motion, the expression for the velocity of sound a is given by,

$$a^2 = d\, p/\eta\ d\ \rho \tag{18}$$

which on using the relation $$d\, p/d\ \rho = \gamma\ p/\rho,$$

gives $$a^2 = \gamma\ p/\rho\ \eta \tag{19}$$

Substituting equation (16) into the expression for η, for the ultra-relativistic case, the velocity of sound $a = 1/\sqrt{3}$. The magneto-relativistic form of the energy equation is given by Taub[5] as

$$\eta - \tfrac{1}{2}\, \{g_{ij} - (g_{ui}\, g_{uj}/g_{uu})\}\ u^i\, u^j = \text{Constant} \tag{20}$$

Using relations (3), (6), (7), (17) and (19), equation (20) can be written in terms of 3-velocity as

$$a^2\, \eta/(\gamma - 1) + \beta^2\, (u^2 + v^2 + w^2)/2 = a^2{}_\infty\ \eta_\infty\ /(\gamma - 1) + \beta^2{}_\infty\ U^2/2 \tag{21}$$

where a quantity with suffix ∞ denotes the value of that quantity in the undisturbed state and U is the corresponding free stream velocity. Using (18), equations (12) and (13) are combined to give (in two dimensional steady flow),

$$\frac{\beta\Omega}{a^2_r} = \left(\beta \frac{\partial u_\alpha}{\partial x_\alpha} + u_\alpha \frac{\partial \beta}{\partial x_\alpha}\right) - \left(u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y}\right)^{-1} \left(\beta \frac{\partial u_\alpha}{\partial x_\alpha} + u_\alpha \frac{\partial \beta}{\partial x_\alpha}\right) \times$$
$$\times \left[\mu \beta^2 \Omega H_\theta H^\theta + \frac{3}{2} \mu \left\{ u (H_\theta H^\theta)_{,1} + v (H_\theta H^\theta)_{;2} \right\}\right] \tag{22}$$

where $\Omega = u^2 \dfrac{\partial u}{\partial x} + u\, v \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) + v^2 \dfrac{\partial v}{\partial y}$, $a^2_r = a^2/\beta^2$ and α, θ take values 1 and 2 only.

Let Ψ denote a potential function such that

$$u_\alpha = \frac{\partial \Psi}{\partial x_\alpha} = \Psi_{x\alpha}$$

o that we have

$$u = \frac{\partial \Psi}{\partial x} = \Psi_x, \quad \frac{\partial u}{\partial x} = \Psi_{xx}$$
$$v = \frac{\partial \Psi}{\partial y} = \Psi_y, \quad \frac{\partial v}{\partial y} = \Psi_{yy}$$

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and then equation (22) becomes

$$\Psi_{xx} (a_r^2 - \Psi_x^2) + \Psi_{yy} (a_r^2 - \Psi_y^2) - 2 \Psi_x \Psi_y \Psi_{xy} + \frac{a_r^2}{\beta} \Psi_{x\alpha} \frac{\partial \beta}{\partial x_\alpha} =$$
$$= \left(\Psi_x \frac{\partial p}{\partial x} + \Psi_y \frac{\partial p}{\partial y}\right)^{-1} \left[\mu\, a_r^2 \left(\Psi_{xx} + \Psi_{yy} + \frac{1}{\beta} \Psi_{x\alpha} \frac{\partial \beta}{\partial x_\alpha}\right)\right] \left[\beta^2 H_\theta H^\theta \times\right.$$
$$\left.\times \left(\Psi_x^2 \Psi_{xx} + 2 \Psi_x \Psi_y \Psi_{xy} + \Psi_y^2 \Psi_{yy}\right) + \frac{3}{2} \Psi_{x\alpha} (H_\theta H^\theta)_{,\alpha}\right] \tag{23}$$

we introduce perturbation potential function ϕ defined by

$$\Psi = U_x + \phi$$

such that,

$$u_\alpha = \Psi_{x\alpha} = (Ux)_{x\alpha} + \phi_{x\alpha}$$
$$u = \Psi_x = U + \phi_x = U + u'$$
$$v = \Psi_y = \phi_y = v'$$

Let the perturbation h in the field Γ be given by

$$\Gamma = H_\infty x + h$$

so that

$$H_1 = \frac{\partial}{\partial x}(\Gamma) = H_\infty + h_x, \quad H_2 = \frac{\partial}{\partial y}(\Gamma) = h_y$$
$$H^2 = H_i H^i = (H_\infty + h_x)^2 + h_y^2$$

Taking the two dimensional form of equation (21) and remembering that $a_r^2 = a^2/\beta^2$, equation (23) can be written as

$$\phi_{xx} \left[1 - \beta^2 \left(\frac{\eta}{\eta_\infty}\right) \left(\frac{U}{a_\infty}\right)^2\right] + \phi_{yy} - \frac{\beta^2}{a^2_\infty} \phi_{xx} \left[\left(\frac{\gamma-1}{\eta_\infty} + \frac{2\eta}{\eta_\infty}\right) U \phi_x + \right.$$
$$\left. + \left(\frac{\gamma-1}{2\eta_\infty} + \frac{\eta}{\eta_\infty}\right) \phi_x^2 + \frac{\gamma-1}{2\eta_\infty} \left(1 - \frac{\beta^2_\infty}{\beta^2}\right) U^2\right] - \frac{\beta^2 \phi_{yy}}{a^2_\infty} \left[\frac{\gamma-1}{\eta_\infty} U \phi_x + \frac{\gamma-1}{2\eta_\infty} \phi^2_x + \right.$$

$$+\left(\frac{\gamma-1}{2\eta_\infty}+\frac{\eta}{\eta_\infty}\right)\phi^2_y+\frac{\gamma-1}{2\eta_\infty}\left(1-\frac{\beta^2_\infty}{\beta^2}\right)U^2\Big]-\frac{2\beta^2}{a^2_\infty}\frac{\eta}{\eta_\infty}\left(U+\phi_x\right)\phi_y\,\phi_{xy}+$$

$$+\frac{a^2_r\,\beta\,\eta}{a^2_\infty\,\eta_\infty}\Big[(Ux)_{x\alpha}+\phi_{x\alpha}\Big]\frac{\partial\beta}{\partial x_\alpha}=\left[\left(U+\phi_x\right)\frac{\partial p}{\partial x}+\phi_y\frac{\partial p}{\partial y}\right]^{-1}\Big[\frac{\eta}{\eta_\infty}\frac{a^2}{a^2_\infty}$$

$$\left\{\phi_{xx}+\phi_{yy}+\frac{1}{\beta}\left((Ux)_{x\alpha}+\phi_{x\alpha}\right)\frac{\partial\beta}{\partial x_\alpha}\right\}\Big]\;\Big[\mu\,\beta^2\left\{(H_\infty+h_x)^2+h^2_y\right\}\Big\{(U+\phi_x)\,\phi_{xx}+$$

$$+2\,(U+\phi_x)\,\phi_y\,\phi_{xy}+\phi^2_y\,\phi_{yy}\Big\}+3\mu\left\{(U+\phi_x)(H_\infty+h_x)(h_{xx}+h_{xy})+\phi_y\,h_y\,(h_{yy}+h_{xy})\right\}\Big] \quad (24)$$

(24) is the equation governing the flow of compressible magneto-relativistic gas and is suitable for finding out small amplitude disturbances in subsonic, supersonic and transonic regions. We treat below these three cases separately.

SUBSONIC MAGNETO-RELATIVISTIC FLOW

We linearize equation (24) for the subsonic magneto-relativistic flow assuming that the local values o p, ρ and e vary very slightly from their values p_∞, ρ_∞, e_∞ in the undisturbed state, i.e. $p = p_\infty + p'$, $\rho = \rho_\infty + \rho'$ etc., where the primed quantities denote small amplitude disturbances from their free stream conditions. The power and cross products of the velocity terms and their gradients, terms related with magnetic field and their gradients are neglected and ϕ_x/U, ϕ_y/U, h_x/H_∞, $h_y/H_\infty << 1$. Neglecting higher order terms of small quantities and their derivatives, we observe that $V^2 \rightarrow U^2$, $\eta \rightarrow \eta_\infty$ and equation (24) then takes the form

$$\left[1-\left(\frac{U}{a_{r\infty}}\right)^2\right]\phi_{xx}+\phi_{yy}=0 \quad (25)$$

where $a^2_{r\infty} = a^2_\infty/\beta^2_\infty$. Since for subsonic magneto-relativistic flow $U/a_{r\infty} < 1$, equation (25) shows that the flow is of elliptic type. Introducing the affine transformations,

$$x=\delta\,\xi,\quad y=\zeta,\quad \delta^2=1-\left(\frac{U}{a_{r\infty}}\right)^2 \quad (26)$$

In equation (25) for characterizing the flow of the fluid, we get

$$\phi_{\xi\xi}+\phi_{\zeta\zeta}=0 \quad (27)$$

which is the Laplace equation in two dimensions. Now analysing equation (12), we determine the expression for pressure coefficient in the case of small amplitude disturbances. For two dimensional steady flow, the linearized x component of equation (12) (neglecting terms of order higher than two) is given by

$$\frac{\partial p'}{\partial x}=-\rho_\infty\,\eta_\infty\,\beta^2_\infty\,U\frac{\partial u'}{\partial x}+\mu\,\beta^2_\infty\,H^2_\infty\,U\frac{\partial u'}{\partial x}+3\,\mu\,H_\infty\,h_{xx} \quad (28)$$

which on integration gives

$$p-p_\infty=-\rho_\infty\,\eta_\infty\,\beta^2_\infty\,U\,\phi_x+\mu\,\beta^2_\infty\,H^2_\infty\,U\,\phi_x+3\,\mu\,H_\infty\,h_x \quad (29)$$

Again using the affine transformation given by (26), equation (29) becomes

$$p-p_\infty=-\rho_\infty\,\eta_\infty\,\beta^2_\infty\,U\,\phi_\xi/\delta+\mu\,\beta^2_\infty\,H^2_\infty\,U\,\phi_\xi/\delta+3\,\mu\,H_\infty\,h_\xi/\delta \quad (30)$$

Let the pressure coefficient P_r, in the case of compressible magneto-relativistic flow, be defined by

$$P_r=\frac{2}{\rho_\infty U^2}\left(p-p_\infty\right) \quad (31)$$

which on substituting the value of $p - p_\infty$ from (30) and neglecting higher order terms becomes

$$P_r=\frac{P_i\,\eta_\infty\,\beta^2_\infty}{\delta}\left(1-\frac{b^2_\infty}{\eta_\infty}\right) \quad (32)$$

where $P_i = -2\phi_\xi/U$ is the local incompressible pressure coefficient corresponding to the flow described by (27) and b_∞ is the Alfven speed for the undisturbed gas. Substituting the values of δ and β^2_∞ in equation (32) and simplifying we have

$$\frac{P_r}{\eta_\infty P_i} = \frac{1 - \dfrac{b^2_\infty}{\eta_\infty}}{\left[(1-U^2)\ (1-U^2-U^2/a^2_\infty)\right]^{\frac{1}{2}}} \tag{33}$$

For the general compressible flow, it will be convenient to consider $na_\infty = 1$ (the velocity of light c has been taken to be unity) and then (33) becomes

$$\frac{P_r}{\eta_\infty P_i} = \left(1 - b^2_\infty/\eta_\infty\right)\left[\left\{1-\left(\frac{M_\infty}{n}\right)^2\right\}\left\{1-\left(\frac{M_\infty}{n}\right)^2 - M^2_\infty\right\}\right]^{-\frac{1}{2}} \tag{34}$$

where M_∞ is the free stream Mach number. In the case when the flow is free from relativistic and magnetic effects i.e. when $\eta_\infty \to 1$, $M_\infty/n \to 0$ and $b_\infty \to 0$, the equation (34) reduces to the well known Prandtl Glauert law[6].

SUPERSONIC MAGNETO-RELATIVISTIC FLOW

For small amplitude disturbances in the case of supersonic magneto-relativistic flow we start with equation (24) which on neglecting higher order terms of small quantities and their derivatives, as in the last section, transforms to equation (25). But, since for supersonic magneto-relativistic flow $U/a_{r\infty} > 1$ the equation (25) is of hyperbolic type. Using equation (29) and (31), pressure coefficient is given by

$$P_r = \frac{2}{\rho_\infty U^2}\left[-\rho_\infty\ \eta_\infty\ \beta^2_\infty\ U\ \phi_x + \mu\ \beta^2_\infty\ H_\infty\ U\ \phi_x\right] \tag{35}$$

The pressure coefficient $-2\phi_x/U$, in this case is related to the local flow deflection θ as $-2\phi_x/U = 2\theta/\alpha$, analogous to the procedure adopted by Liepmann & Roshko[6], where $\alpha^2 = \left(\dfrac{U}{a_{r\infty}}\right)^2 - 1$. Equation (35) can then be written as

$$\frac{P_r}{2\,\theta\,\eta_\infty} = \left(1 - b^2_\infty/\eta_\infty\right)\left[\left\{1-\left(\frac{M_\infty}{n}\right)^2\right\}\left\{M^2_\infty + \left(\frac{M_\infty}{n}\right)^2 - 1\right\}\right]^{-\frac{1}{2}} \tag{36}$$

In the case of non-relativistic flow having no magnetic effects, equation (36) reduces to the well known Ackeret solution[6].

TRANSONIC MAGNETO-RELATIVISTIC FLOW

For the transonic magneto-relativistic case the decrease in the third term relative to the first term on the left hand side of (24) must be taken into account. It is assumed that the streamwise gradients of ϕ, u' and h are comparatively greater than the lateral gradients of potential or velocity components or field strength components. Thus in this case the first three terms are retained in (24) with $\eta \to \eta_\infty$ and $\beta^2_\infty = 1/(1-U^2)^{\frac{1}{2}}$. Equation (24) then reduces to

$$\left[1-\left(\frac{U}{a_{r\infty}}\right)^2\right]\phi_{xx} - \left[\frac{(\gamma-1)+2\,\eta_\infty}{U\eta_\infty}\right]\left[\frac{U}{a_{r\infty}}\right]^2 \phi_x\,\phi_{xx} + \phi_{yy} = 0 \tag{37}$$

In view of the subsonic and supersonic flows, the condition taken for transonic flow is $U/a_{r\infty} = 1$. Thus following the method of Spreiter[7], equation (37) may be written as

$$\phi_{yy} = \frac{[(\gamma-1)+2\,\eta_\infty]\,\phi_x\,\phi_{xx}}{U\,\eta_\infty} = k_r\,\phi_x\,\phi_{xx} \tag{38}$$

Considering the boundary conditions it is easy to see that the flow is tangential to the locally perturbed streamline and ϕ_x and ϕ_y vanish at infinity. The former condition, as in linearised theory, is approximated to

$$\left(\phi_y\right)_{Streamline} = U\left(\frac{dy}{dx}\right)$$

Taking into account the case of local supersonic flow, in which $\phi_x = u' > 0$ and dropping the primes of the perturbed quantities, equation (38) reduces to

$$- \lambda \phi_{xx} + \phi_{yy} = 0 \qquad (\lambda > 0) \tag{39}$$

where $\lambda = k_r u$. Assuming the variation of λ to be very small for the initial stages of analysis, it may be written as constant. The solution for u on the locally perturbed streamline is

$$u = - U \lambda^{-\frac{1}{2}} \frac{dy}{dx} \tag{40}$$

which on differentiation gives

$$\frac{du}{dx} = - U \lambda^{-\frac{1}{2}} \frac{d^2y}{dx^2} \tag{41}$$

Substituting the value of λ into (41) and using (29) and (31) we get

$$P_r = \left[\frac{2 \eta_\infty \beta^2_\infty}{U} \left(b^2_\infty / \eta_\infty - 1 \right) \right] \left[C - \frac{3 U^{3/2} \eta_\infty^{\frac{1}{2}}}{2 \left[(\gamma - 1) + 2 \eta_\infty \right]^{\frac{1}{2}}} \right]^{\frac{2}{3}} \tag{42}$$

where C is the constant of integration and $\theta (= dy/dx)$ is the local flow deflection. For applying this result to the locally magneto-relativistic supersonic flow, the constant C is evaluated from (40) from which it is clear that u and thus P_r is zero at $\theta = 0$. Thus

$$P_r = \beta^2_\infty \left(b^2_\infty - \eta_\infty \right) \left[18 \eta_\infty / \left\{ (\gamma - 1) + 2 \eta_\infty \right\} \right]^{\frac{1}{3}} \theta^{\frac{2}{3}} \qquad \theta < 0 \tag{43}$$

Considering the transonic flow condition $U/a_{r\infty} = 1$ and $n a_\infty = 1$ we have

$$\beta^2_\infty = 1 + \frac{1}{n^2} \tag{44}$$

Using the approximation $2\eta_\infty >> (\gamma - 1)$ and (44), equation (43) reduces to the expression

$$\frac{P_r}{\eta_\infty} = \left(b^2_\infty / \eta_\infty - 1 \right) \left(1 + \frac{1}{n^2} \right) \left(9 \theta^2 \right)^{\frac{1}{3}} \tag{45}$$

For the non magneto-relativistic flow, where $\eta_\infty \rightarrow 1$, $\frac{1}{n^2} \rightarrow 0$ and $b_\infty \rightarrow 0$, equation (45) reduces to the ordinary transonic expression originally obtained by Spreiter & Alksne[7].

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