INTEGRAL TRANSFORM AND HEAT CONDUCTION IN A HOLLOW CONE WITH RADIATION

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A new integral transform is developed whose kernel is a spherical function, a solution of Legendre differential equation. This transform is used to determine the temperature at any point in a hollow finite cone whose inner angle is α and outer angle is β, with boundary conditions of radiation type on the outside and inside surfaces having independent radiation constants. It is evident that most of the possible problems on boundary conditions in hollow cones can be solved by particularising the method described here.

The purpose is to solve a problem of finding temperature inside a hollow finite cone bounded by surfaces $\theta = \alpha$ and $\theta = \beta$ when there is heat radiation on its outside and inside surfaces. For this purpose, we introduce first a new integral transform[1] on lines with that of Marchi and Zgrablich.

Consider the Legendre differential equation of order ν.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu+1)y = 0 \tag{1}$$

The kernel of the integral transform is the general solution of (1) with boundary conditions given in (3). We shall further use this transform to solve the physical problem stated above. It is presumed that all functions involved satisfy Dirichlet's conditions.

THE TRANSFORM AND INVERSION

Let us seek the solution of our Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu+1)y = 0 \tag{2}$$

for the boundary conditions

$$y(a) + k_1 y'(a) = 0 \quad ; \quad y(b) + k_2 y'(b) = 0 \tag{3}$$

where k_1, and k_2 are independent radiation constants.

The general solution of (2) is

$$y = C_1 P_\nu(x) + C_2 Q_\nu(x) \tag{4}$$

where $P_\nu(x)$ and $Q_\nu(x)$ are Legendre functions of the first and second kind, respectively. Substituting (4) in (3), we get

$$\left.\begin{aligned} C_1 P_\nu(a) + C_2 Q_\nu(a) + k_1 [C_1 P'_\nu(a) + C_2 Q'_\nu(a)] = 0 \\ C_1 P_\nu(b) + C_2 Q_\nu(b) + k_2 [C_1 P'_\nu(b) + C_2 Q'_\nu(b)] = 0 \end{aligned}\right\} \tag{5}$$

Let

$$\left.\begin{aligned} P_\nu(k_i, x) = P_\nu(x) + k_i P'_\nu(x) \\ Q_\nu(k_i, x) = Q_\nu(x) + k_i Q'_\nu(x) \end{aligned} \quad \text{for } (i = 1, 2)\right\} \tag{6}$$

Then (5) can be rewritten as

$$\left.\begin{aligned} C_1 P_\nu(k_1, a) + C_2 Q_\nu(k_1, a) = 0 \\ C_1 P_\nu(k_2, b) + C_2 Q_\nu(k_2, b) = 0 \end{aligned}\right\} \tag{7}$$

Hence

$$-\frac{C_2}{C_1}=\frac{P_\nu(k_1,a)}{Q_\nu(k_1,a)}=\frac{P_\nu(k_2,b)}{Q_\nu(k_2,b)}$$

Therefore

$$P_\nu(k_1,a)\,Q_\nu(k_2,b)-P_\nu(k_2,b)\,Q_\nu(k_1,a)=0 \tag{8}$$

Let ν_n be the root of (8), then the general solution takes the form

$$y_{1,n}(x)=\frac{C_1}{Q_{\nu_n}(k_1,a)}\left[\,P_{\nu_n}(x)\,.\,Q_{\nu_n}(k_1,a)-Q_{\nu_n}(x)\,P_{\nu_n}(k_1,a)\,\right] \tag{9}$$

$$y_{2,n}(x)=\frac{C_1}{Q_{\nu_n}(k_2,b)}\left[\,P_{\nu_n}(x)\,Q_{\nu_n}(k_2,b)-Q_{\nu_n}(x)\,P_{\nu_n}(k_2,b)\,\right] \tag{10}$$

By a linear combination of (9) and (10), we get the general solution

$$y_n(x)=E_{\nu_n}(k_1,k_2,x)=P_{\nu_n}(x)\,[\,Q_{\nu_n}(k_1,a)+Q_{\nu_n}(k_2,b)\,]-$$
$$-Q_{\nu_n}(x)\,[\,P_{\nu_n}(k_1,a)+P_{\nu_n}(k_2,b)\,] \tag{11}$$

which are solutions of Legendre differential equation (2), of order ν, and satisfy the boundary conditions (3). Such functions are Eigen functions[2] and are orthogonal in the interval $(a\ ,\ b)$. Now let us define the finite integral transform

$$\bar{f}(n)=\int_a^b f(x).\,E_{\nu_n}(k_1,k_2,x)\,dx \tag{12}$$

where $\bar{f}(n)$ is the transform of $f(x)$ with respect to the kernel $E_{\nu_n}(k_1,k_2,x)$

Inversion Theorem

If $f(x)$ satisfies Dirichlet's conditions in the interval $a \leqslant x \leqslant b$ and $\bar{f}(n)$ exists, then

$$f(x)=\sum_n a_n\,E_{\nu_n}(k_1,\ k_2,\ x) \tag{13}$$

where

$$a_n=\frac{\bar{f}(n)}{C_n}\quad\text{and}\quad C_n=\int_a^b[\,E_{\nu_n}(k_1,k_2,x)\,]^2\,dx \tag{14}$$

Proof

Let us take that $f(x)$ is expressable in the form

$$f(x)=\sum_i a_i\,E_{\nu_i}(k_1,k_2,x)$$

Taking the transform of both the sides we get

$$\bar{f}(n)=\sum_i a_i\int_a^b E_{\nu_i}(k_1,k_2,x)\,E_{\nu_n}(k_1,k_2,x)\,dx$$

From the property of orthogonality

$$\int_a^b E_{\nu_i}(k_1,k_2,x).\,E_{\nu_n}(k_1,k_2,x)\,dx=0,\qquad \text{for } i\neq n$$

Hence

$$\bar{f}(n) = a_n \int_a^b \Big[E_{\nu_n}(k_1, k_2, x) \Big]^2 dx,$$

therefore $a_n = \bar{f}(n) / C_n$, where C_n is given by (14).

CALCULATION OF C_n

If $\omega_\nu(z)$ and $\omega_\sigma(z)$ denote any solutions of the Legendre's differential equation[3] with parameters ν and σ respectively, then from second relation[3] [p 169, 3·12, (1)]

$$\int_c^d \omega_\nu \cdot \omega_\sigma \, dz = \frac{1}{(\nu-\sigma)(\nu+\sigma+1)} \cdot \Big[z(\nu-\sigma)\,\omega_\nu \cdot \omega_\sigma + \sigma\omega_\nu\,\omega_{\sigma-1} - \nu \cdot \omega_{\nu-1} \cdot \omega_\sigma \Big]_c^d \quad (15)$$

then

$$\int_c^d \omega_\sigma \cdot \omega_\sigma \, dz = \frac{1}{2\sigma+1} \Big[z\,\omega^2_\sigma + \lim_{\nu\to\sigma} \frac{\sigma\omega_\nu \cdot \omega_{\sigma-1} - \nu\omega_{\nu-1} \cdot \omega_\sigma}{\nu-\sigma} \Big]_c^d$$

Changing ν to $\sigma + h$ and taking the limit $h \to 0$

$$\int_c^d \omega^2_\sigma \, d\sigma = \frac{1}{2\sigma+1} \Big[z\,\omega^2_\sigma - \omega_{\sigma-1} \cdot \omega_\sigma + \sigma L \Big] \quad (16)$$

where

$$L = \lim_{h\to 0} \frac{\omega_{\sigma+h} \cdot \omega_{\sigma-1} - \omega_{\sigma+h-1} \cdot \omega_\sigma}{h} \quad (17)$$

As stated ω_σ is the solution of

$$(1-z^2)\frac{d^2\omega_\sigma}{dz^2} - 2z\frac{d\omega_\sigma}{dz} + \sigma(\sigma+1)\,\omega_\sigma = 0 \quad (18)$$

Let $\omega'_\sigma = \dfrac{d\omega_\sigma}{d\sigma}$. Differentiate (18) with respect to σ we get

$$(1-z^2)\frac{d^2\omega'_\sigma}{dz^2} - 2z\frac{d\omega'_\sigma}{dz} + \sigma(\sigma+1)\,\omega'_\sigma + (2\sigma+1)\,\omega_\sigma = 0 \quad (19)$$

Further $\omega_{\sigma+h}$ will be the solution of the differential equation

$$(1-z^2)\frac{d^2\omega_{\sigma+h}}{dz^2} - 2z \cdot \frac{d\omega_{\sigma+h}}{dz} + (\sigma+h)(\sigma+h+1)\,\omega_{\sigma+h} = 0 \quad (20)$$

Now for small h assuming the approximation

$$\omega_{\sigma+h} = \omega_\sigma + h\omega'_\sigma \quad (21)$$

and substituting this in (20), we get

$$(1-z^2)\left\{\frac{d^2\omega_\sigma}{dz^2} + h\frac{d^2\omega'_\sigma}{dz^2}\right\} - 2z\left\{\frac{d\omega_\sigma}{dz} + h\frac{d\omega'_\sigma}{dz}\right\} +$$
$$+ (\sigma+h)(\sigma+h+1)\,\omega_\sigma + h(\sigma+h)(\sigma+h+1)\,\omega'_\sigma = 0$$

Now substituting values from (18) and (19) we calculate the first order approximation of ω'_σ

$$\omega'_\sigma \approx -\frac{\omega_\sigma}{2\sigma+h+1}$$

Hence

$$\omega_{\sigma+h} \approx \frac{2\sigma+1}{2\sigma+h+1}\,\omega_\sigma \tag{22}$$

Hence substituting value of $\omega_{\sigma+h}$ and $\omega_{\sigma+h-1}$ in (17), we get

$$L = \lim_{h\to 0} \frac{\omega_{\sigma-1}\cdot\omega_\sigma}{h}\left[\frac{2\sigma+1}{2\sigma+h+1} - \frac{2\sigma-1}{2\sigma+h-1}\right]$$

$$= \frac{2\omega_{\sigma-1}\cdot\omega_\sigma}{4\sigma^2-1}$$

Substituting this in (16), we get

$$\int_c^d (\omega_\sigma)^2\,dz = \frac{1}{2\sigma+1}\left[z\omega^2_\sigma - \frac{\omega_\sigma\,\omega_{\sigma-1}}{4\sigma^2-1}\,(4\sigma^2-2\sigma-1)\right]_c^d$$

Further changing z to x and taking limits from a to b, ω_σ being replaced by $E_{\nu_n}(k_1, k_2, x)$, we get

$$C_n = \int_a^b \left[E_{\nu_n}(k_1, k_2, x)\right]^2 dx$$

$$= \frac{1}{2\nu_n+1}\left[x\,E^2_{\nu_n} - \frac{E_{\nu_n}\cdot E_{\nu_n-1}}{4\nu_n^2-1}\,(4\nu_n^2-2\nu_n-1)\right]_a^b \tag{23}$$

PROPERTIES OF THE TRANSFORM

We investigate the effect of this transform on the expression $(1-x^2)\dfrac{d^2f}{dx^2} - 2x\dfrac{df}{dx}$. Taking the transform of this expression and integrating by parts we obtain

$$\int_a^b \left[(1-x^2)\frac{d^2f}{dx^2} - 2x\frac{df}{dx}\right] E_{\nu_n}(k_1, k_2, x)\,dx$$

$$= \left[(1-x^2)\,E_{\nu_n}\left\{\frac{df}{dx} - \frac{\frac{dE_{\nu_n}}{dx}}{E_{\nu_n}}\,f\right\}\right]_a^b + \int_a^b \left[(1-x^2)\frac{d^2E_{\nu_n}}{dx^2} - 2x\frac{dE_{\nu_n}}{dx}\right] f\,.\,dx \tag{24}$$

It can be easily deduced from (11) and (6), taking into account (8), that

$$\left.\frac{\frac{dE_{\nu_n}}{dx}}{E_{\nu_n}}\right]_{x=b} = -\frac{1}{k_2} \quad\text{and}\quad \left.\frac{\frac{dE_{\nu n}}{dx}}{E_{\nu n}}\right]_{x=a} = -\frac{1}{k_1} \tag{25}$$

Now substituting these values in (24), we get

$$\int_a^b \left[(1-x^2)\frac{d^2f}{dx^2} - 2x\frac{df}{dx}\right] E_{\nu_n}(k_1, k_2, x)\,dx$$

$$= \frac{1-b^2}{k_2}\,E_{\nu_n}(k_1, k_2, b)\left\{f + k_2\frac{df}{dx}\right\}_{x=b} - \frac{1-a^2}{k_1}\,E_{\nu_n}(k_1, k_2, a)\left\{f + k_1\frac{df}{dx}\right\}_{x=a} -$$

$$-\nu_n(\nu_n+1)\bar{f}(n) \tag{26}$$

This is the basic property of our transform that will enable us to solve problems in which such terms are involved.

TRANSFORM OF SOME FUNCTIONS

We shall obtain here the integral transforms, defined in (12) of some functions $f(x)$, defined in the range $-1 < x < 1$. We choose a and b, the limits of the integration of the integral transform, such that $-1 < a < b < 1$.

Let $f(x) = P_m(x)$, then

$$\bar{f}(n) = \int_a^b P_m(x) \cdot E_{\nu_n}(k_1, k_2, x)\, dx \tag{27}$$

Making use of first relation[3]

$$\bar{f}(n) = \frac{1}{(m-\nu_n)(m+\nu_n+1)} \left[(1-x^2) \left\{ P_m(x) \frac{dE_{\nu_n}}{dx} - E_{\nu_n} \frac{dP_m(x)}{dx} \right\} \right]_a^b \tag{28}$$

Further replacing $\dfrac{dP_m(x)}{dx}$ by using the recurrence relation[3] [p. 161, 3·8 (19)], (28) can be rewritten as

$$\bar{f}(n) = \frac{1}{(m-\nu_n)(m+\nu_n+1)} \left[(1-x^2) P_m(x) \frac{dE_{\nu_n}}{dx} + mE_{\nu_n} \left\{ xP_m(x) - P_{m-1}(x) \right\} \right]_a^b$$

Now applying the property given by (25), we get

$$\bar{f}(n) = \frac{E_{\nu_n}(k_1, k_2, b)}{(m-\nu_n)(m+\nu_n+1)} \left[m P_{m-1}(b) - \left\{ \frac{1-b^2}{k_2} + mb \right\} P_m(b) \right] -$$
$$- \frac{E_{\nu_n}(k_1, k_2, a)}{(m-\nu_n)(m+\nu_n+1)} \left[m P_{m-1}(a) - \left\{ \frac{1-a^2}{k_1} + ma \right\} P_m(a) \right] \tag{29}$$

Further any function $f(x)$ defined in the range $-1 < x < 1$, if satisfies Dirichlet's conditions, then it can be expanded in a series of Legendre polynomials.

$$f(x) = \sum_{m=0}^{\infty} B_m P_m(x) \tag{30}$$

where

$$B_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) \cdot P_m(x)\, dx \tag{31}$$

Since the range of integration $(a, b,)$ of the transform, is a subinterval of $(-1, 1)$ hence the expansion (30) is valid in the said range. Assuming the validity of the term by term integration the transform of $f(x)$, defined by (12) will be

$$\bar{f}(n) = \sum_{m=0}^{\infty} B_m \int_a^b P_m(x) \cdot E_{\nu_n}(k_1, k_2, x)\, dx$$

$$= \sum_{m=0}^{\infty} \frac{B_m}{(m-\nu_n)(m+\nu_n+1)} \left[E_{\nu_n}(k_1, k_2, b) \left\{ m P_{m-1}(b) - \left(\frac{1-b^2}{k_2} + mb \right) \cdot \right.\right.$$
$$\left.\left. P_m(b) \right\} - E_{\nu_n}(k_1, k_2, a) \left\{ mP_{m-1}(a) - \left(\frac{1-a^2}{k_1} + ma \right) P_m(a) \right\} \right] \tag{32}$$

where B_m is given by relation (31) which can be easily determined.

APPLICATION TO STEADY STATE PROBLEM OF A HOLLOW FINITE CONE WITH RADIATION

Consider the steady state solution of a hollow finite cone, whose axis is coincident with z-axis, defined by $\alpha \leqslant \theta \leqslant \beta$, $0 \leqslant r \leqslant a$, where α and β are interior and exterior angles of the cone, and (r, θ, Φ) are spherical coordinates, with boundary conditions of radiation type. Assuming symmetry with respect to z-axis, the temperature $u(r, \theta)$ at any point of the cone will be the solution of the equation :

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial u}{\partial \theta}\right] = 0 \tag{33}$$

If we take $p = \cos\theta$ and $p_1 = \cos\beta$ and $p_2 = \cos\alpha$, then $u(r, p)$ is the solution of

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + \frac{\partial}{\partial p}\left[(1-p^2)\frac{\partial u}{\partial p}\right] = 0 \tag{34}$$

Let the boundary conditions be

$$\left.\begin{aligned} u(r, p_1) + k_1 u_p(r, p_1) &= r^\rho \\ u(r, p_2) + k_2 u_p(r, p_2) &= r^\sigma \end{aligned} \quad 0 \leqslant r \leqslant a \right\} \tag{35}$$

where k_1 and k_2 are radiation constants, and

$$u(a, p) = f(p) = P_m(p) \tag{36}$$

Now applying the transform (12) with respect to p to (34) and (36) and denoting $\bar{u} = \bar{u}(r, n)$ as the transform of $u(r, p)$ and $\bar{f}(n)$ as the transform of $f(p)$, we get

$$r^2 \frac{d^2 \bar{u}}{dr^2} + 2r\frac{d\bar{u}}{dr} - \nu_n(\nu_n + 1)\bar{u} + \phi(r) = 0 \tag{37}$$

where

$$\phi(r) = \frac{1-p_2^2}{k_2} E_{\nu_n}(k_1, k_2, p_2)\, r^\sigma - \frac{1-p_1^2}{k_1} E_{\nu_n}(k_1, k_2, p_1)\, r^\rho \tag{38}$$

$$\bar{u}(a, n) = \bar{f}(n) \tag{39}$$

where $\bar{f}(n)$ is given by (29)

Equation (37) is second order equation of homogeneous type, calculating the complementary function and the particular integral, the solution can be written as

$$\bar{u}(r, n) = A r^{\nu_n} + B r^{-\nu_n - 1} - \frac{1-p_2^2}{k_2} E_{\nu_n}(k_1, k_2, p_2) \frac{r^\sigma}{(\sigma - \nu_n)(\sigma + \nu_n + 1)} + $$

$$+ \frac{1-p_1^2}{k_1} E_{\nu_n}(k_1, k_2, p_1) \frac{r_1^\rho}{(\rho - \nu_n)(\rho + \nu_n + 1)} \tag{40}$$

Since at $r = 0$, $\bar{u}$ has to be finite, therefore $B = 0$ and by applying boundary condition (39), we get

$$A = \frac{1}{a^{\nu_n}}\left[\bar{f}(n) + \frac{1-p_2^2}{k_2} E_{\nu_n}(k_1, k_2, p_2) \frac{a^\sigma}{(\sigma - \nu_n)(\sigma + \nu_n + 1)} - \right.$$

$$\left. - \frac{1-p_1^2}{k_1} E_{\nu_n}(k_1, k_2, p_1) \frac{a^\rho}{(\rho - \nu_n)(\rho + \nu_n + 1)}\right] \tag{41}$$

Hence

$$\bar{u}(r, n) = A r^{\nu_n} - \frac{1-p_2^2}{k_2} E_{\nu_n}(k_1, k_2, p_2) \frac{r^\sigma}{(\sigma - \nu_n)(\sigma + \nu_n + 1)} + $$

$$+ \frac{1-p_1^2}{k_1} E_{\nu_n}(k_1, k_2, p_1) \frac{r^\rho}{(\rho - \nu_n)(\rho + \nu_n + 1)} \tag{42}$$

where A is given by (41)

Now applying inversion formula (13) to (42) we get the solution

$$u(r, p) = \sum_n \frac{1}{C_n} E_{\nu_n}(k_1, k_2, p)\, \bar{u}(r, n) \tag{43}$$

where C_n is given by (14) and $\bar{u}(r, n)$ by (42)

APPLICATION TO THE TRANSIENT HEAT EQUATION IN A HOLLOW FINITE CONE WITH RADIATION

We consider the transient heat flow in a hollow finite cone, with boundary conditions of radiation type. Let the conductivity of the material be k_0, a constant and $\frac{\rho c}{k_0} = \frac{1}{m^2}$, where ρ is the density and c is the specific heat of the material of the cone. The temperature $u(r, p, t)$, where $p = \cos\theta$, is the solution of the heat equation :

$$\frac{1}{m^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial p}\left[(1-p^2)\frac{\partial u}{\partial p}\right]. \tag{44}$$

The boundary conditions are

$$\left.\begin{aligned} u(r, p_1, t) + k_1 u_p(r, p_1, t) &= R_1(r, t) \\ u(r, p_2, t) + k_2 u_p(r, p_2, t) &= R_2(r, t) \end{aligned}\right\} \text{for } t > 0,\ 0 \leqslant r \leqslant a \tag{45}$$

where $u_p = \frac{\partial u}{\partial p}$

Assuming that the solution is of the type

$$u = A\, e^{-\lambda^2 t}\, T(r, p) \tag{46}$$

$$\text{and } R_1(r, t) = Ae^{-\lambda^2 t} f_1(r) \text{ and } R_2(r, t) = Ae^{-\lambda^2 t} f_2(r)$$

The equations (44) and (45) reduces to

$$-\frac{\lambda^2}{m^2} T(r, p) = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial p}\left[(1-p^2)\frac{\partial T}{\partial p}\right] \tag{47}$$

and

$$\left.\begin{aligned} T(r, p_1) + k_1 Tp(r, p_1) &= f_1(r) \\ T(r, p_2) + k_2 Tp(r, p_2) &= f_2(r) \end{aligned}\right\} \tag{48}$$

Now taking $\frac{\lambda^2}{m^2} = k^2$, the equation (47) can be rewritten as

$$r^2 \frac{\partial^2 T}{\partial r^2} + 2r\frac{\partial T}{\partial r} + k^2 T + \frac{\partial}{\partial p}\left[(1-p^2)\frac{\partial T}{\partial p}\right] = 0 \tag{49}$$

Naylor[4] has developed a Lebedev transform for the group of terms $r^2\Psi_{rr} + 2r\Psi_r - k^2 r^2 \Psi$ and has suggested that a transform can be devised for group of terms $r^2\Psi_{rr} + 2r\Psi_r + k^2 r^2 \Psi$. Such a transform can be easily developed as

$$\bar{T}(v, p) = \int_0^a h(v, r)\, T(r, p)\, r^{-\frac{1}{2}}\, dr \tag{50}$$

where

$$h(v, r) = J_v(kr)\, Y_v(ka) - J_v(ka)\, Y_v(kr) \tag{51}$$

and its inversion formula will be

$$T(r, p) = \frac{1}{2i\, r^{\frac{1}{2}}} \int_L v\, \bar{T}(v, p)\, \frac{J_v(kr)}{J_v(ka)}\, dv \tag{52}$$

where L is the path $Re\,(v) = c,\ 0 < c < r$, the strip in which $\overline{T}\,(v, p)$ is regular.

Applying the transform (50) to (49) and (48), we get

$$(v^2 - \tfrac{1}{4})\,\overline{T}\,(v, p) + \frac{2a^{\frac{1}{2}}}{\pi}\,T\,(a, p) + \frac{\partial}{\partial p}\left[\,(1 - p^2)\,\frac{\partial \overline{T}}{\partial p}\right] = 0 \tag{53}$$

and

$$\left.\begin{aligned} \overline{T}\,(v, p_1) + k_1\,\overline{T}\,(v, p_1) &= F_1\,(v) \\ \overline{T}\,(v, p_2) + k_2\,\overline{T}\,(v, p_2) &= F_2\,(v) \end{aligned}\right\} \tag{54}$$

Now applying the transform defined by (12), we get

$$(v^2 - \tfrac{1}{4})\,\bar{\tau}\,(v, n) + \frac{2a^{\frac{1}{2}}}{\pi}\ \tau\,(a, n) - \nu_n\,(\nu_n + 1)\ \bar{\tau}\,(v, n) +$$

$$+ \left\{ \frac{1 - p_2{}^2}{k_2}\,E_{\nu_n}\,(k_1, k_2, p_2)\,F_2\,(v) - \frac{1 - p_1{}^2}{k_1}\,E_{\nu_n}\,(k_1, k_2, p_1)\,F_1\,(v) \right\} = 0 \tag{55}$$

where $\bar{\tau}\,(v, n)$ and $\tau\,(v, n)$ is the transform (12) of $\overline{T}\,(v, p)$ and $T\,(v, p)$.

Writing $$W(v) = \left\{ \frac{1 - p_2{}^2}{k_2}\,E_{\nu_n}\,(k_1, k_2, p_2)\,F_2\,(v) - \frac{1 - p_1{}^2}{k_1}\,E_{\nu_n}\,(k_1, k_2, p_1)\,F_1\,(v) \right\}$$

we get

$$\bar{\tau}\,(v, n) = -\,\frac{W(v) + \dfrac{2a^{\frac{1}{2}}}{\pi}\ \tau\,(a, n)}{(v^2 - \frac{1}{4}) - \nu_n\,(\nu_n + 1)} \tag{56}$$

Hence taking the inversion step by step

$$\overline{T}\,(v, p) = \sum_n \frac{1}{C_n}\ \bar{\tau}\,(v, n)\,E_{\nu_n}\,(k_1, k_2, p) \tag{57}$$

Where C_n is given by (14)

Further by applying (52)

$$T\,(r, p) = \frac{1}{2i\,r^{\frac{1}{2}}}\int_L v\,\overline{T}\,(v, p)\,\frac{J_v\,(kr)}{J_v\,(ka)}\,dv$$

$$= \frac{1}{2\,ir^{\frac{1}{2}}}\int_L v\ \frac{J_v\,(kr)}{J_v\,(ka)}\left[\sum_n \frac{1}{C_n}\ \bar{\tau}\,(v, n)\,E_{\nu_n}\,(k_1, k_2, p)\right] dv$$

$$T\,(r, p) = \frac{1}{2i\,r^{\frac{1}{2}}}\sum_n \frac{E_{\nu_n}\,(k_1, k_2, p)}{C_n}\int_L v\ \frac{J_v\,(kr)}{J_v\,(ka)}\ \bar{\tau}\,(v, n)\,dv \tag{58}$$

Further the arbitrary constant A in (46) can be easily determined by prescribed initial conditions.

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