VIBRATIONS OF CIRCULAR CYLINDERS OF A PERFECTLY CONDUCTING ELASTIC MATERIAL

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The problems of radial vibrations of a long circular solid cylinder with a transverse magnetic field and rotary vibrations of a hollow cylinder with radial magnetic field are solved. The results of the case of an infinite medium with a cylindrical cavity are given. The frequency equation in each case, is solved in particular cases approximately.

In this paper the problems of radial vibrations of a solid circular cylinder with a transverse magnetic field and rotary vibrations of a hollow circular cylinder with radial magnetic field are solved. The frequency equations are solved approximately. The results of an infinite medium with a cylindrical cavity are given as particular cases. In the problem of radial vibrations, the frequency equation gives two independent modes in the first approximation for copper and aluminium. In the second problem, the frequency has a set of equal and opposite real roots and the other set is imaginary.

RADIAL VIBRATIONS OF A SOLID CYLINDER

Let us consider the radial vibrations of a solid circular cylinder in the presence of an applied transverse magnetic field. In the case of radial vibrations, the displacement is given to be

$$\vec{u} = u(r)\, e^{ipt}\, \vec{i_r}$$

and the applied magnetic field is $H = H_0\, \vec{i_\theta}$ where $\vec{i_r}$, $\vec{i_\theta}$, $\vec{i_z}$ are unit vectors in cylindrical coordinate system. The electric and secondary magnetic fields are given by[1]

$$\vec{e} = -H_0\, ipu\, e^{ipt}\, \vec{i_z}$$

$$\vec{h} = -H_0\, (du/dr)\, e^{ipt}\, \vec{i_\theta}.$$

The equations of motion give[1],

$$(\lambda + 2\mu + H_0^2)\frac{d^2 u}{dr^2} + \frac{\lambda + 2\mu}{r}\;\frac{du}{dr} + \left[p^2 \rho - \frac{\lambda + 2\mu}{r^2}\right] u = 0 \tag{1}$$

Using the notation

$$\alpha = \frac{H_0^2}{2(\lambda + 2\mu + H_0^2)}$$

$$\beta^2 = \frac{\rho a^2 p^2 (1 - 2\alpha)}{\lambda + 2\mu}$$

the solution of (1) that is finite for $r = 0$, is

$$u(r) = A\, a^{1-\alpha}\, r^{\alpha}\, J_{1-\alpha}\frac{\beta r}{a}$$

where a is the radius of the cylinder and A is an arbitrary constant. It is to be observed that (i) α and β are dimensionless, (ii) α lies between 0 and 0·5, and (iii) $\alpha = 0$ corresponds to the case of purely elastic material.

The boundary conditions give[1], for $k \approx 1$

$$J_{1-\alpha}(\beta) - \frac{\beta J_{2-\alpha}(\beta)(1-\sigma)}{1 - 2\alpha\sigma} = 0 \tag{2}$$

where σ is the Poisson's ratio of the medium. H_0 is the magnetic field that prevails outside the cylinder because the magnetic permeability of the medium is taken to be unity approximately.

When only first terms are taken in the corresponding Bessel function expansions, equation (2) gives

$$\beta = \left(\frac{2(2-\alpha)(1-2\alpha\sigma)}{1-\sigma}\right)^{1/2}$$

and when only the first two terms are taken, we get

$$\beta^4 - \beta^2\left[4(3-\alpha) + \frac{2(3-\alpha)(1-2\alpha\sigma)}{1-\sigma}\right] + \frac{8(3-\alpha)(2-\alpha)(1-2\alpha\sigma)}{1-\sigma} = 0$$

Values of β are computed for two values of σ (copper and aluminium). (see Table 1)

TABLE 1

Values of β for two values of σ (Cu = 0·3 and Aluminium = 0.378)

$\alpha \backslash \sigma$	Cu (0·34)	Al (0·378)
0·0	$\pm 4\cdot092508627$ $\pm 2\cdot083814565$	$\pm 4\cdot195626599$ $\pm 2\cdot010726160$
0·1	$\pm 3\cdot983390185$ $\pm 1\cdot980632643$	$\pm 3\cdot981171758$ $\pm 1\cdot978643544$
0·2	$\pm 3\cdot875018426$ $\pm 1\cdot874871008$	$\pm 3\cdot825725363$ $\pm 1\cdot824017786$

A Particular Case

Radial vibrations of an infinite medium with a cylindrical cavity : As in ref. 2 the problem can be solved for the general time dependence using Laplace Transformations with respect to time variable. The displacement satisfies the same equation of motion (8). The solution, that is finite as $r \to \infty$, is

$$u = B\, a^{1-a}\, r^{a-1}\, J_{a-1}\, (\beta r/a)$$

or

$$u = B\, a^{1-a}\, r^{a-1}\, K_{1-a}\, (\beta r/a)$$

where B is a constant of integration and a is the radius of the cavity. The expressions for the stresses etc., are obtained by replacing J_{1-a} by K_{1-a} when the second expression is taken for u.

ROTARY VIBRATIONS OF A HOLLOW CYLINDER

Let the internal and external radii be denoted by a and b. In this case, the displacement is given by

$$\vec{u} = u(r)\, e^{ipt}\, \vec{i}_\theta \qquad (3)$$

The electric field and the secondary magnetic field are given by

$$\vec{e} = -\, ip\, H_0\, e^{ipt}\, u\, \vec{i}_z \qquad (4)$$

$$\vec{h} = H_0\, e^{ipt}\, (du/dr)\, \vec{i}_\theta .$$

The equations of motion give

$$\frac{d^2u}{dr^2} + \frac{\mu}{H_0^2 + \mu}\, \frac{du}{dr} + \left[\frac{p^2\rho}{\mu + H_0^2} - \frac{\mu}{\mu + H_0^2)}\, \frac{1}{r^2}\right] u = 0 \qquad (5)$$

Let

$$\beta^2 = \frac{\rho\, a^2\, p^2}{\mu}\, (1 - 2\alpha) \text{ and } \alpha = \frac{H_0^2}{2(\mu + H_0^2)}$$

Then the solution of (5) is

$$u = a^{1-\alpha}\, r^{\alpha}\, A\, J_{1-\alpha}(\beta r/a) + B\, J_{\alpha - 1}(\beta r/a)$$

where A and B are constants of integration. Here α and β are dimensionless and α lies between 0 and 0·5. The results of purely elastic cylinder can be obtained by putting $\alpha = 0$.

The boundary conditions give[1], for $K \approx 1$

$$A[2\alpha J_{1-\alpha}(\beta) - \beta\, J_{2-\alpha}(\beta)] + B\, [(2\alpha - 1)\, 2\alpha J_{\alpha-1}(\beta) - 2\beta\alpha\, J_\alpha\, (\beta)$$
$$- 2(1-\alpha)\,(1-2\alpha)\, J_{\alpha-1}(\beta) - \beta\,(1-2\alpha)\, J_\alpha\, (\beta)] = 0 \qquad (6)$$

and

$$A\, [2a\alpha\, J_{1-\alpha}(\beta b/a) - \beta b\, J_{2-\alpha}(\beta b/a) + B\, [2a\alpha\, (2\alpha - 1)\, J_{\alpha-1}(\beta b/a) - 2\beta b\alpha\, J_\alpha(\beta b/a)$$
$$- 2a\,(1-\alpha)\,(1-2\alpha)\, J_{\alpha-1}(\beta b/a) - \beta b\,(1-2\alpha)\, J_\alpha\, (\beta b/a)] = 0 \qquad (7)$$

Eliminating A and B from (6) and (7), we get the frequency equation. For the first order approximation by taking the first terms in the expansion of Bessel functions and $b = 2a$, the values of β obtained from the above equations are

for $\alpha = 0\cdot1$ $\quad \beta = \pm\, 0\cdot3283008320, \quad \pm\, 0\cdot7510689468i;$

for $\alpha = 0\cdot2$ $\quad \beta = \pm\, 0\cdot4153984030, \quad \pm\, 1\cdot000707250i;$

Corresponding to $\alpha = 0$, i.e. for purely elastic material of the order of approximation considered here, we get $\beta = 0$. Here one pair of values of β is imaginary. Correspondingly, there will not be any vibrations. Hence there is only one mode of vibration whereas in the case of purely elastic hollow cylinder, the periodic vibrations do not exist.

REFERENCE

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