A NOTE ON
TEMPERATURE DISTRIBUTION OF A VISCOUS INCOMPRESSIBLE FLUID IN A CIRCULAR PIPE UNDER PERIODIC RATE OF HEAT GENERATION

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(*Received* 23 *March* 1971 ; *revised* 7 *August* 1971)

An expression for the temperature distribution in a circular pipe is obtained when viscous incompressible fluid is flowing through it. Effect of viscous dissipation is not neglected and the rate of heat generation per unit volume varies periodically with time.

The attention of many research workers in fluid dynamics has been diverted towards the study of heat transfer in viscous incompressible fluids in the recent years. The interest in this subject has been developed by a paper of Bhatnagar & Tikekar[1] in which they have discussed the temperature distribution of a viscous incompressible fluid flowing in a channel bounded by two co-axial circular cylinders with the rate of heat generation per unit volume as an exponential function of time. They have obtained the expression by neglecting the effect of viscous dissipation.

The temperature distribution of a viscous incompressible fluid flowing in a circular pipe with the rate of heat generation per unit volume varying periodically with time is discussed in the present investigation. Effect of viscous dissipation is not neglected. The expression for the temperature thus obtained has been compared with that of Lal[2] where he has obtained the expression by neglecting the dissipation term. The expression for the temperature is then obtained in dimensionless form. This consists of two parts, the one varies periodically with time and the other is the transient part which vanishes in the limit as t tends to infinity. It is also seen that the contribution of the transient part is insignificant when $t > 2$.

Here the expression for the temperature distribution is obtained with the conditions that the surface $r' = r'_0$ (*i*) has zero initial temperature, and (*ii*) is always being kept at zero temperature.

In the present note the velocity distribution is steady while the temperature distribution is unsteady. The temperature distribution does not affect the flow field of an incompressible fluid with constant properties. We have taken a fluid having these properties.

FORMULATION AND SOLUTION OF THE PROBLEM

We assume that the temperature T' of the liquid is independent of its axial position x', then the energy equation[3] in the present case reduces to

$$\rho' c' \frac{\partial T'}{\partial t'} = \frac{\partial Q'}{\partial t'} + k' \left(\frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'} \frac{\partial T'}{\partial r'} \right) + \mu' \left(\frac{\partial u'}{\partial r'} \right)^2, \tag{1}$$

where $\left(\frac{\partial u'}{\partial r'} \right)^2$ is the energy dissipation function, μ' is the coefficient of viscosity of the fluid, $\frac{\partial Q'}{\partial t'}$ is the rate of heat generation per unit volume in the fluid, c' and k' are respectively the specific heat and the coefficient of heat conductivity of the fluid. Here dashes denote dimensional quantities.

We make equation (1) dimensionless by introducing

$$y = \frac{r'}{r_0'}, \quad u = \frac{u'}{u_m'}, \quad t = \frac{\nu' t'}{r_0'^2}, \quad T = \frac{T'}{\theta'}, \quad f(t) = \frac{\nu' Q'}{k' \theta'}, \tag{2}$$

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where u_m' is the maximum velocity in the pipe, $\nu' \left(= \frac{\mu'}{\rho'}\right)$ is the kinematic viscosity, $y \left(= \frac{r'}{r_0'}\right)$ is the non-dimensional radial coordinate, $u \left(= \frac{u'}{u_m'}\right)$ is the non-dimensional axial velocity, θ' is a characteristic temperature, $f(t)$ is a function of time only such that

$$\left(\frac{\partial f}{\partial t}\right)_{t=0} = 0.$$

Equation (1) then becomes

$$\sigma \frac{\partial T}{\partial t} = \frac{\partial f}{\partial t} + \left(\frac{\partial^2 T}{\partial y^2} + \frac{1}{y}\frac{\partial T}{\partial y}\right) + \frac{1}{4}\beta\sigma\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

where β is the non-dimensional constant $\frac{4u_m'^2}{\theta' c'}$ and σ is the Prandtl number.

In the present case

$$u' = u_m'\left(1 - \frac{r'^2}{r_0'^2}\right). \tag{4}$$

Introducing the non-dimensional variables in (4), we get

$$u = (1 - y^2). \tag{5}$$

From (3) and (5), we get

$$\sigma \frac{\partial T}{\partial t} = \frac{\partial f}{\partial t} + \left(\frac{\partial^2 T}{\partial y^2} + \frac{1}{y}\frac{\partial T}{\partial y}\right) + \beta\sigma y^2. \tag{6}$$

We now assume that the rate of heat generation per unit volume in the fluid varies periodically with time. Hence we take

$$\frac{\partial f}{\partial t} = \sin t. \tag{7}$$

Equation (6) then becomes

$$\sigma \frac{\partial T}{\partial t} = \sin t + \left(\frac{\partial^2 T}{\partial y^2} + \frac{1}{y}\frac{\partial T}{\partial y}\right) + \beta\sigma y^2. \tag{8}$$

Let $\overline{T} = \int_0^\infty e^{-st}\, T dt$ be the Laplace transform of T and let T_0 be the initial value of T.

Multiplying equation (8) by e^{-st} and integrating between the limits zero to infinity, we get

$$\frac{\partial^2 \overline{T}}{\partial y^2} + \frac{1}{y}\frac{\partial \overline{T}}{\partial y} - p^2 \overline{T} = -\left[\sigma T_0 + \frac{1}{1+s^2} + \frac{\beta\sigma y^2}{s}\right], \tag{9}$$

where $p^2 = \sigma s$.

Now we shall find T_0.

Initially the rate of heat generation is zero and the temperature is steady in the channel. Hence

$$\frac{d^2 T_0}{dy^2} + \frac{1}{y}\frac{dT_0}{dy} = -\beta\sigma y^2. \tag{10}$$

The solution of (10) under the boundary conditions

$$T_0 = \text{finite when } y = 0,$$

and $$T_0 = 0 \quad \text{when } y = 1$$

is $$T_0 = \frac{\beta\sigma}{16}\,(1 - y^4).$$

Substituting this value of T_0 in (9), we get

$$\frac{\partial^2 \bar{T}}{\partial y^2} + \frac{1}{y}\,\frac{\partial \bar{T}}{\partial y} - p^2 \bar{T} = -\left[\frac{\beta\sigma^2}{16}\,(1 - y^4) + \frac{1}{1 + s^2} + \frac{\beta\sigma y^2}{s}\right] \tag{11}$$

The solution (11) under the boundary conditions

$$\bar{T} = \text{finite when } y = 0\,,$$

and $$\bar{T} = 0 \quad \text{when } y = 1$$

is $$\bar{T} = \frac{\beta\sigma}{16}\left(\frac{1 - y^4}{s}\right) + \frac{1}{\sigma s(1 + s^2)}\left[1 - \frac{I_0(py)}{I_0(p)}\right] \tag{12}$$

Now applying Laplace inversion theorem, we get

$$T = \frac{\beta\sigma}{16}\,(1 - y^4) + \frac{\cos t}{\sigma}\left[\frac{\text{ber}\, y\sqrt{\sigma}.\ \text{ber}\sqrt{\sigma} + \text{bei}\, y\sqrt{\sigma}.\ \text{bei}\sqrt{\sigma}}{\text{ber}^2\sqrt{\sigma} + \text{bei}^2\sqrt{\sigma}} - 1\right] +$$

$$+ \frac{\sin t}{\sigma}\left[\frac{\text{ber}\, y\sqrt{\sigma}.\ \text{bei}\sqrt{\sigma} - \text{bei}\, y\sqrt{\sigma}.\ \text{ber}\sqrt{\sigma}}{\text{ber}^2\sqrt{\sigma} + \text{bei}^2\sqrt{\sigma}}\right] +$$

$$+ \frac{2}{\sigma}\sum_{n=1}^{\infty} \frac{J_0(\alpha_n y)}{\alpha_n(1 + \alpha_n^4\,\sigma^{-2})\,J_1(\alpha_n)}\,.\,e^{-\alpha_n^2 t/\sigma} \tag{13}$$

where α_n are the positive roots of $J_0(\alpha) = 0$, and

$$\text{ber}\, y\sqrt{\sigma} = \sum_{l=0}^{\infty} \frac{(-1)^l\,(\frac{1}{2}\,y\sqrt{\sigma})^{4l}}{(2l\,!)^2},\ \text{bei}\, y\sqrt{\sigma} = \sum_{l=0}^{\infty} \frac{(-1)^l\,(\frac{1}{2}\,y\sqrt{\sigma})^{2+4l}}{\{(2l+1)!\}^2}$$

are known as Thompson functions.

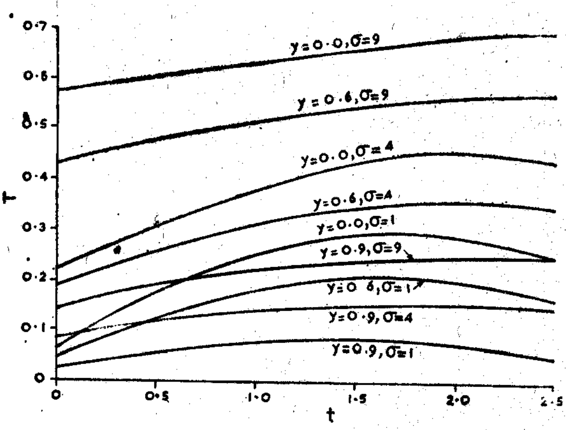

Fig 1—Variation of T with t for $\sigma = 1,\ 4,\ 9$ and $\beta = 1$

Fig. 1 for fixed y ($y = 0{\cdot}0,\ 0{\cdot}6,\ 0{\cdot}9$) showing the variation of T with t have been drawn for $\sigma = 1$, $\beta = 1$; $\sigma = 4$, $\beta = 1$ and $\sigma = 9$, $\beta = 1$. Here we have used the first five terms of the transient part and the tables[4] for Thompson functions for the numerical work.

We find that the transient part is very small compared to the periodic part when $t > 2$, hence the transient part is insignificant and T varies periodically with t in this range. From Fig. 1 it is obvious that the temperature at any point inside the pipe increases with the increase of σ. The increase of temperature with the increase of β is also obvious from (13).

Expression (13) for the temperature does not agree with Lal[2]. His expression for the temperature with the present boundary conditions does not contain the first and the last terms because he has solved the energy equation by neglecting the dissipation term and by assuming

$$\frac{1}{\rho c}\,\frac{\partial Q}{\partial t} = a e^{int}, \qquad T = T(r) e^{int},$$

whereas in our case the dissipation term has not been neglected. Expression (13) for the temperature is in complete agreement with similar result obtained by Ballabh[5] where he has obtained the expression for the velocity by using the method of superposability.

Hence (13) is a more general solution of (6) and has been confirmed by Laplace transform method.

ACKNOWLEDGEMENT

The authors are grateful to Professor P. L. Bhatnagar for his constant encouragement.

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