RENEWAL FUNCTION FOR ANY ARBITRARY PERIOD—A BAYESIAN WAY

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The usual format on which the consumption data and activity for spare parts of a system are available, makes it difficult for the evaluation of the renewal function on the procedure as laid down by Cox[1]. The author has, therefore, discussed in this paper a model through Bayesian Approach as to how in such cases the renewal function could be obtained for any period of activity, provided the past experience are available in the form of a sample information. For a Beta-Prior density for the mean demand rate of the Poisson demand distribution, the renewal function in its analytical and asymptotic forms are obtained.

The limitation to the procedure as outlined by Cox[1] for the evaluation of the renewal function $H(t)$ is well-known. Thus, instead of the analytical form of the distribution of the number of renewals N_t in a fixed period, usually the asymptotic expression for $H(t)$ is obtained. Moreover, the usual format on which the consumption data and activity are available, makes the above procedure all the more intractable. Therefore, a conditional probability approach was suggested by Youngs *et. al.*[2] that the probability of demand for an item in a future period would be obtained provided the demands for the same item during the past period were made known.

Now, that the consumption of aircraft spare parts is characteristically low, the prediction of the true mean demand for an item is usually restricted. The author[3] in his paper, therefore, had made the assumption that the uncertainty about the mean demand rate could be defined by a Beta-Prior density as an alternative to the assumption of a Gamma-Prior as indicated by Youngs *et. al*,[2]. The Bayesian rule was then used by the author[3] to arrive at a revised probability density for the Poisson Parameter of the Poisson demand distribution, based on the past experience available from the sample information, and the renewal function for an unit period of operation was finally obtained in its analytical as well as asymptotic forms.

In this present paper, the author, however, has made an attempt to obtain the similar expressions for the renewal function for any arbitrary period of activity, knowing the sample information on the consumption of spares for r-successive period, each of equal length of operation, but other than the period of activity for which the renewal function is obtained.

DEMAND MODEL

Let the uncertainty about the mean demand rate λ be defined by a Prior-density as

$$f(\lambda) = \frac{\lambda^{\nu_1 - 1}(1-\lambda)^{\nu_2 - 1}}{B(\nu_1,\nu_2)} \tag{1}$$

where $\nu_1, \nu_2 > 0$, $B(\cdot,\cdot)$ is a complete Beta-function and $0 < \lambda < 1$. The estimates of ν_1 and ν_2 are given by the author[3].

For aircraft spare parts, Youngs *et. al.*[2] observed that the demand for an item belonging to the aircraft system, closely follows a Poisson distribution; therefore, the Probability $P(X)$ that the demand for an item drawn from a population of items governed by $f(\lambda)$ will be X in any period of activity, say T units of flying hours, is given as

$$P(X=x) = \int \frac{e^{-\lambda T} \quad (\lambda T)^x}{x!} f(\lambda)\, d\lambda, \text{ where } X = 0, 1, 2, \ldots \ldots$$

Combining the above equation with equation (1), one obtains

$$P(X = x) = \frac{B(x + \nu_1, \nu_2) \cdot T^x}{B(\nu_1, \nu_2) \cdot x!} \, {}_1F_1(x + \nu_1; x + \nu_1 + \nu_2; -T) \tag{2}$$

where ${}_1F_1(\,.\,;\,.\,;-T)$ is a confluent hypergeometric function[4]

Now, if the number of demands $\{x_i\}$ where $i = 1, 2, \ldots, r$, for a particular spare part are observed in r-successive periods each of equal length ($= T$ flying hours) of activity, one obtains the revised value for $f(\lambda)$ using the Bayes rule[5] as the posterior density for λ under the conditions that the demands for particular item in the r successive periods of experience are independent and identically distributed as Poisson and that the demand rate for the item remains same throughout the periods. Thus, one has

$$f^*(\lambda / X = \{x_i\}) = \frac{e^{-r\lambda T} (\lambda T)^{S_r + \nu_1 - 1} (1-\lambda)^{\nu_2 - 1}}{B(S_r + \nu_1, \nu_2) \cdot {}_1F_1(S_r + \nu_1; S_r + \nu_1 + \nu_2; -\gamma T)} \tag{3}$$

where

$$\sum_{i=1}^{r} x_i = S_r.$$

Therefore, the predictive demand distribution of an item in the period for the $(r+1)^{th}$ class of operational period of duration L flying hours, given the past experience of the sample information of $\{x_i\}$ demands for the same item, is obtained as

$$P(Y = y / X = \{x_i\}) = \frac{B(S_r + \nu_1 + y, \nu_2) \, y! \, {}_1F_1(S_r + \nu_1 + y; S_r + \nu_1 + \nu_2 + y; -rT - L)}{B(S_r + \nu_1, \nu_2) \cdot y! \, {}_1F_1(S_r + \nu_1; S_r + \nu_1 + \nu_2; -rT)} \tag{4}$$

THE RENEWAL FUNCTION

The probability generating function of the Predictive demand distribution as given in equation (4), is obtained as follows :

$$G(Z/X = \{x_i\}) = \sum_{y=0}^{\infty} Z^y P(Y/X) = \sum_{y=0}^{\infty} \frac{(S_r + \nu_1)_y (ZL)^y \, {}_1F_1(T_r + \nu_1 + y; S_r + \nu_1 + {}_2 + y; -rT - L)}{(S_r + \nu_1 + \nu_2)_y \, (y)! \; {}_1F_1(S_r + \nu_1; S_r + \nu_1 + \nu_2; -rT)}$$

where

$$(S_r + \nu_1)_y = (S_r + \nu_1)(S_r + \nu_1 + 1) \ldots (S_r + \nu_1 + y - 1).$$

Applying, the multiplication theorem of ${}_1F_1(\,.\,;\,.\,;\,.\,)$, the above $P.\,g.\,f.$ reduces to

$$G(Z/X = \{x_i\}) = \frac{{}_1F_1(S_r + \nu_1; S_r + \nu_1 + \nu_2; -rT - L + ZL)}{{}_1F_1(S_r + \nu_1; S_r + \nu_1 + \nu_2; -rT)} \tag{5}$$

The renewal function for a period of L flying hours of activity, is, therefore, obtained as

$$H_{rT, rT+L}(L) = \lim_{z \downarrow 1} G'(Z/X = \{x_i\})$$

where G' is the first derivative of G w.r.t. Z

or

$$H_{rT, rT+L}(L) = \frac{(S_r + \nu_1) L \, {}_1F_1(S_r + \nu_1 + 1; S_r + \nu_1 + \nu_2 + 1; -rT)}{(S_r + \nu_1 + \nu_2) \, {}_1F_1(S_r + \nu_1; S_r + \nu_1 + \nu_2; -rT)} \tag{6}$$

THE ASYMPTOTIC VALUE OF THE RENEWAL FUNCTION

If $\hat{\lambda}$ be the true mean demand rate of the Poisson demand distribution, then for large number of successive intervals of operation r of duration T units each, one gets through the Weak Law of Large number[6] that

$$\lim_{r \uparrow \infty} \left(\frac{S_r}{r} \right) \doteq \hat{\lambda}\, T$$

Now, using the limiting value for ${}_1F_1$ (. ; . ; .), one finds

$$\lim_{r \uparrow \infty} G\,(Z/X = \{x_i\}) = e^{-\hat{\lambda} L (1 - Z)} \tag{7}$$

and this is the p.g.f. of the Poisson demand distribution with the mean demand as $\hat{\lambda}\, L$ for the predictive period. Thus, the limiting value of the renewal function for an item is the Product of the true mean demand rate and the length of the Predictive period.

EXAMPLE

Let the stocking policy for the spare parts of a system be ten-quarter period and the number of demands for the first four-quarter of an item is observed to be zero. It is now required to find the expected number of renewals to be required for the remaining six-quarter of the period for the same item.

Assume that the demand pattern for such item follows a Poisson distribution with an unknown demand rate which is defined by a Beta-Prior density. The parameters ν_1 and ν_2 are assumed to be respectively 0.2 and 0.5 (say) obtained on the lines given by the author[3]. Now, the renewal function for the predictive six-quarter period of operation, is obtained from equation (6) as under

$$H\,(L = \text{Six=quarter}) \quad \frac{0{\cdot}2 \times 0{\cdot}6\; {}_1F_1\,(1{\cdot}2;\, 1{\cdot}7;\, -0{\cdot}4)}{0{\cdot}9\; {}_1F_1\,(0{\cdot}2;\, 0{\cdot}7;\, -0{\cdot}4)}$$

Applying, the recurrence formula[4] for ${}_1F_1$ (. ; . ; $-$ 0.4), the above equation reduces to

$$H\,(L = \text{Six-quarter}) = 1.1238.$$

CONCLUSION

The expressions for the renewal function in its analytical and asymptotic forms are reduced to the results obtained by the author[3] by taking T and L equal to unity. The present result illustrates that even if one has the experience of the demand behaviour for an item for any unit of operation in the past, he can easily obtain the expected number of renewals for the same item for any future period.

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