START-UP FLOW WITH VELOCITY SLIP IN THE ENTRANCE REGION OF A POROUS CIRCULAR TUBE

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Suddenly started laminar flow of an incompressible viscous fluid in the entrance region of a porous circular tube is studied analytically, taking into account the tangential velocity slip at its surface. Adopting a similarity velocity profile of the form of *n*th. degree parabola in the boundary layer, the integrated forms of the equations of continuity, momentum and energy are solved by the method of characteristics. The entry length is seen reduced by the fluid injection across the wall of the tube and increased by the tangential velocity slip at it. Both these factors are found increasing the start-up time required for the full development of the flow.

The problem of entry length flow in an impermeable circular tube has been widely studied. A discussion of earlier work is given in Campbell and Slattery[1] and more recently in Fargie and Martin[2]. In their own analysis Campbell and Slattery considered macrospic mechanical energy balance across the entire cross-section in order to take into consideration the energy loss due to viscous dissipation in the increasingly thickening boundary layer. The suddenly started flow in the entrance region of a circular tube has been studied by Avula[3] and by Avula and Young[4], respectively assuming a steady and a time-dependent inlet velocity, by Schiller's method of integrating the equation of boundary layer flow with the help of an assumed parabolic profile and using the Euler's equation for inviscid fluid flow in the core region. Noblesse and Farell[5] have followed the method of Campbell and Slattery[1] in their study of unsteady non-uniform flow in the entrance region of an impermeable circular tube.

Injection of fluid across the walls of channels and pipes is known to reduce the entry length. This fact may be utilised in obtaining fully developed flow both in industrial designs and in laboratory experiments. Horton and Yuan[6] investigated steady flow in the entrance region of a porous walled channel employing Karman-Pohlhausen integral method and assuming both similar as well as non-similar velocity profiles. Gupta[7] reconsidered the problem by the method of Campbell and Slattery[1]. On the other hand, Bansal and Jain[8] followed Schlichting[9] in dividing the entrance region into two zones, considering the one near the inlet as the zone of boundary layer growth and adopting in the farther zone the method of progressive deviation from the parabolic velocity profile.

In all these and other investigations of flow in the entrance region of porous walled channels and tubes the boundary condition of 'no slip' has been applied to the tangential component of velocity. But the experimental evidence indicates a tangential velocity slip on account of the smoothening of the boundary by the fluid trapped in its pores and also owing to the coupled flow taking place inside the porous medium. The empirical boundary condition proposed by Beavers and Joseph[10] has been supported and simplified on statistical considerations by Saffman[11]. The phenomenon of velocity slip has also been examined by Taylor[12] and Richardson[13] and has been supported by further experiments. Sparrow, Beavers and Hung[14] applied the simplified boundary condition to steady flow in a porous walled tube and channel.

The object of the analysis presented here is to consider the effect of velocity slip on the start-up flow in the entrance region of a porous circular tube. On adopting the boundary conditions[13] as in reference (14), the flow inside the porous medium of the wall of the tube is not coupled to the main flow in the tube. Still, it may be added that wall is assumed sufficiently thick to support a substratum of fluid that gives rise to the tangential velocity slip at the inner surface of the tube. A general *n*th. order parabolic velocity profile has been assumed for the boundary layer, and the continuity, momentum and energy equations have been integrated across the entrie cross section and to dispense altogether with the Euler's equation for inviscid flow. The resulting quasi-linear differential equation has been solved by the method of characteristics. The entry length is defined as the length extending from the inlet to the position where the boundary layers on the wall join on the axis. Results following from the fourth degree velocity profile are presented graphically. Sixth degree profile has also been discussed. A similar study has been made by the authors[15] for start-up entry length flow in a porous walled channel also.

EQUATIONS OF MOTION

Cylindrical polar co-ordinates, such that $\bar{x}$ is measured along the axis of the semi infinite circular tube of radius r_0 and $\bar{r}$ perpendicular to it, are employed, the velocity components in these directions being $\bar{u}$ and $\bar{v}$ respectively, the hydrodynamic fluid pressure being $\bar{p}$. The density ρ and the kinematic viscosity γ are constant. Initially, the entire length of the tube is occupied by stationary fluid which is suddenly set in motion at time $\bar{t}=0$ by a uniform inflow at the tube inlet $\bar{x}=0$ with a constant velocity V. At the same time, the fluid injection begins taking place across the wall of the tube uniformly with a constant velocity v_o in the radial direction. Then $Re=Vr_o/v$ appears as the Reynolds number of the main flow, and $\lambda=v_o r_o/v$ arises as the injection parameter. As pointed out by Schlichting[9], in order to ensure that the flow with blowing at the wall satisfies the simplifying conditions which form the basis of the boundary layer theory, λ is not greater than $\sqrt{Re}$ in magnitude. The following dimensionless variables are introduced.

$$\xi = 1 + \lambda x, \; x = \bar{x}/r_0 \, Re, \; r = \bar{r}/r_0, \; t = \bar{t}\, v_0/r_0, \; u = \bar{u}/V, \; v = v/\bar{v}_0, \; p = \bar{p}/\rho\, v^2$$

The equations of continuity and motion now take the form

$$\partial (u\, r)/\partial \xi + \partial (v\, r)/\partial r = 0 \tag{1}$$

and

$$\partial u/\partial t + u \partial u/\partial \xi + v\, \partial u/\partial r = - \partial p/\partial \xi + (\lambda r)^{-1}\, \partial(r\, \partial u/\partial r)/\partial r. \tag{2}$$

The boundary conditions are

$$\left.\begin{array}{lll} \text{at the entry} & \xi = 1, \; u = 1, \; v = 0 \text{ for } t \geqslant 0 \\ \text{at the axis} & r = 0, \; u = U\,(\xi, t), \; \partial u/\partial r = 0, \; v = 0 \\ \text{at the wall} & r = 1, \; u = u_o\,(\xi, t), \; v = -1. \end{array}\right\} \tag{3}$$

where u_0 is the slip velocity

Following reference (14), the slip velocity at the surface of a saturated porous material is assumed to be given by

$$\bar{u} = (k^{\frac{1}{2}}/\alpha)\, (\partial \bar{u}/\partial\, \bar{n})$$

evaluated at the surface, where k is the permeability of the material of the tube wall, α a dimensionless constant also depending upon the nature of this material only and $\partial \bar{u} / \partial \bar{n}$ is the gradient of the tangential velocity in the direction of the normal to the surface drawn into the fluid. In dimensionless form, the condition is

$$u_o = - \beta\, (\partial u/\partial r)_{r=1} \tag{4}$$

where

$$\beta = k^{\frac{1}{2}} / r_0\, \alpha.$$

In terms of the boundary layer thickness δ, the conditions are

$$\left.\begin{array}{l} \delta = 0 \text{ for } \xi \geqslant 1 \text{ at } t = 0 \\ \text{and} \\ \delta = 0 \text{ at } \xi = 1 \text{ for } t \geqslant 0. \end{array}\right\} \tag{5}$$

SOLUTION

Let the similar velocity profiles in the boundary layer be given by the nth degree parabola

$$\left.\begin{array}{ll} u = U\,[1 - \delta_1 (1 - y/\delta)^n], & 0 \leqslant y \leqslant \delta , \\ u = U , & \delta \leqslant y \leqslant 1 , \end{array}\right\} \quad (6)$$

where $\delta(\xi, t)$ is the boundary layer thickness, $y = 1 - r$ is the distance from the tube wall and $\delta_1 = \delta/(\delta + n\beta)$. Now by integrating (1) with respect to r from the axis to the boundary of the tube, it is found that

$$\int_0^1 u\, r\, dr = \xi - 1/2. \quad (7)$$

Hence, using (6)

$$U = (n+1)(n+2)(\delta + n\beta)(2\xi - 1)/Q, \quad (8)$$

where $Q(\delta) = 2\,\delta^2\,[\delta - (n+2)] + (\delta + n\beta)(n+1)(n+2)$.

Also, by (4), the slip velocity at the tube wall is

$$u_0 = U\, n\beta/(\delta + n\beta). \quad (9)$$

Since, by (7), $\quad (\partial/\partial t) \int_0^1 ur\, dr = 0 ,$

the integral form of the momentum equation (2) is obtained as

$$\partial p/\partial \xi = 2\, u_0\, [1 - (\beta\lambda)^{-1}] - 2\,(\partial/\partial \xi) \int_0^1 u^2\, r\, dr. \quad (10)$$

The equation describing the rate of change of kinetic energy per unit mass is obtained by multiplying each term of (2) by u. Integrating it with respect to r for $0 \leqslant r \leqslant 1$, the equation governing the macroscopic energy balance is found to be

$$(2\xi - 1)(\partial p/\partial \xi) = u_0^2\,[1 - 2/(\lambda\beta)] - (2/\lambda) \int_0^1 (\partial u/\partial r)^2\, r\, dr -$$

$$- (\partial/\partial t) \int_0^1 u^2\, r\, dr - (\partial/\partial \xi) \int_0^1 u^3\, r\, dr. \quad (11)$$

Elimination of $(\partial p/\partial \xi)$ between (10) and (11) yields a quasilinear differential equation

$$P_1\,(\partial\delta/\partial \xi) + P_2\,(\partial \delta/\partial t) = P_3$$

where

$$P_1(\delta, \xi) = (2\,\xi - 1)\, G_1(\delta)$$

$$P_2(\delta) = \delta_1\,(2\,F_2\,F_5 - F_4\,Q)/(2n+1)$$

$$\begin{aligned} P_3(\delta) = {} & [3\,(n+1)(n+2)\,F_1/\{(3n+1)(3n+2)\} - 4\,F_2\,Q]/(2\,n+1) + \\ & + n\beta\, Q\,[2\,Q - n\,\beta\,(n+1)(n+2)] + \\ & + (nQ/\lambda)\,[(n+1)(n+2)\,\{\delta\,(2\,n - \delta) + 2\,\beta\, n\,(2n-1)\}/(2\,n-1) - 2Q] \end{aligned}$$

$$\begin{aligned} F_1(\delta) = {} & \delta^5\,(85\,n^3 + 95\,n^2 + 32\,n + 4) - \delta^4\,(3n+2)\,\{2\,(n+2)(11n^2 + 6n + 1) - \\ & - 3\,\beta\, n\,(3n+1)(7n+2)\} + \delta^3\,(3n+1)(3n+2)\,\{(n+1)(n+2)(2n+1) - \\ & - 6\beta n\,(n+2)(3n+1) + 6\,\beta^2\, n^2\,(2n+1)\} + \beta n\,(n+2)(2n+1)(3\,n+1) \\ & (3n+2) \times [3\,\delta^2\,\{(n+1) - 2\,\beta\, n\} + \beta n\,(3\delta + \beta n)] \end{aligned}$$

$$\begin{aligned} F_2(\delta) = {} & \delta^4\,(7n+2) - 2\,\delta^3\,[(n+2)(3\,n+1) - 2\,\beta n n + 1)] + \\ & + (n+2)(2n+1)\,[\delta^2\,\{(n+1) - 4\,\beta n\} + \beta h\,(n+1)(2\delta + \beta n)] \end{aligned}$$

$$F_3(\delta) = \delta^4 (85n^3 + 95n^2 + 32n + 4) - \delta^3 [(n+2)(3n+2)(11n^2+6n+1) - \beta n (307n^3 + 359n^2 + 128n + 16)] - 2\,\delta^2 \beta n (3n+2)[2(n+2)(11n^2+6n+1) - 3\beta n(3n+1)(7n+2)] - 3\beta^2 n^2 (3n+1)(3n+2)[\delta\{(n+2)(7n+2) - 3\beta n(2n+1)\} + 2\beta n (n+2)(2n+1)]$$

$$F_4(\delta) = \delta^3 (7n+2) - \delta^2 (3n+1)[(n+2) - 6\beta n] - 3\,\delta\beta n[(n+2)(3n+1) - 2\beta n(n+1)] - 4\beta^2 n^2 (n+2)(2n+1)$$

$$F_5(\delta) = 2\,\delta^2 - \delta[(n+2) - 3\beta n] - 2\beta n(n+2)$$

$$G_1(\delta) = -2P_2 + (n+1)(n+2)\,\delta_1 (3F_1 F_5/Q - F_3)/[(2n+1)(3n+1)(3n+2)]$$

The integration of the auxiliary system of ordinary differential equations

$$d\xi/P_1 = dt/P_2 = d\delta/P_3 \qquad (12)$$

gives

$$\xi = \int_0^\delta (P_1/P_3)\, d\delta, \qquad (13)$$

and

$$t = \int_0^\delta (P_2/P_3)\, d\delta. \qquad (1)$$

Connecting ξ and t through δ, the characteristic curves can be drawn in the $\xi - t$ plane, each dividing the plane into two regions. By using the equation (13) or (14) as may be applicable in there spective region of influence of each one of them, the boundary layer thickness may now be determined inversely.

NUMERICAL RESULTS FOR $n = 4$

The system of auxiliary differential equations (12) has been integrated numerically by Runge-Kutta-Gill method, taking the fourth order parabolic profile for velocity in the boundary layer. The effect of velocity slip is brought out by comparing the results corresponding to $\beta = 0.01$ with those for $\beta = 0$ (no-slip) in the two cases of $\lambda = 10$ and 50, corresponding to small and moderately large mass transfer across the tube wall. Fig. 1 shows the boundary layer thickness, δ against the axial variable ξ as given by the equation (13). Since the definition of ξ involves λ, Fig. 2 has been drawn to show the boundary layer thickness explicitly

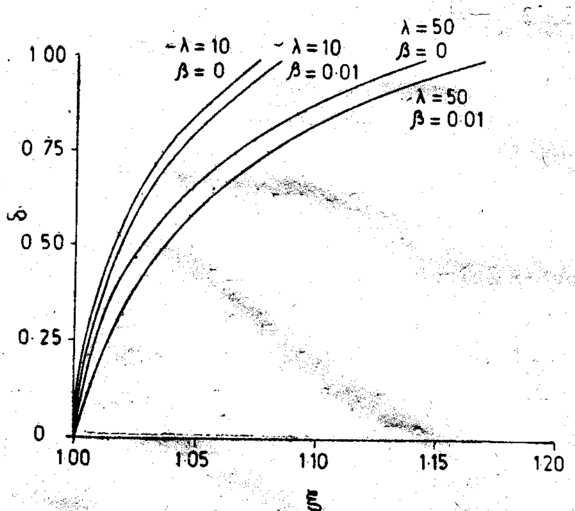

Fig. 1—Boundary layer thickness against ξ (4th degree profile).

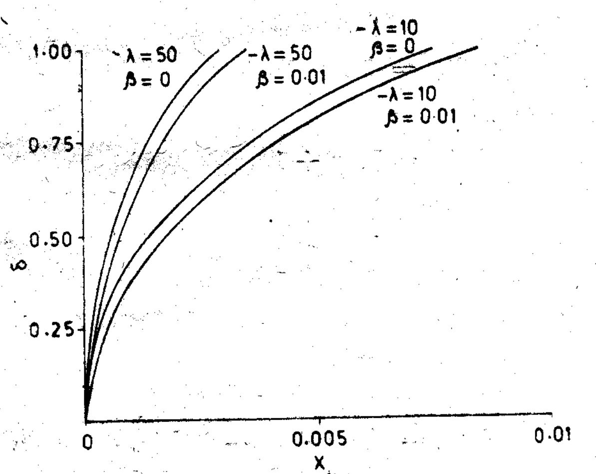

Fig. 2—Boundary layer thickness against X (4th degree profile).

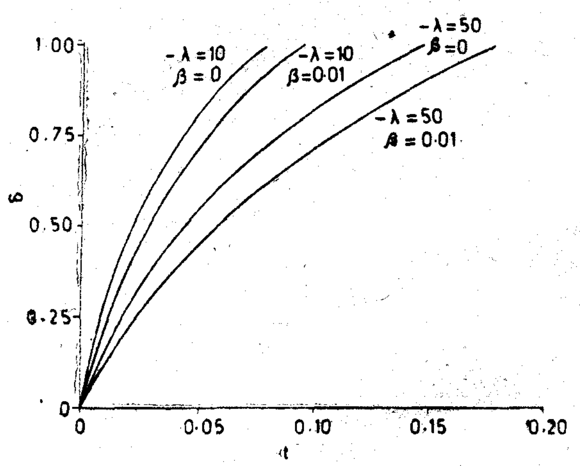

Fig. 3—Boundary layer thickness against time (4th degree profile).

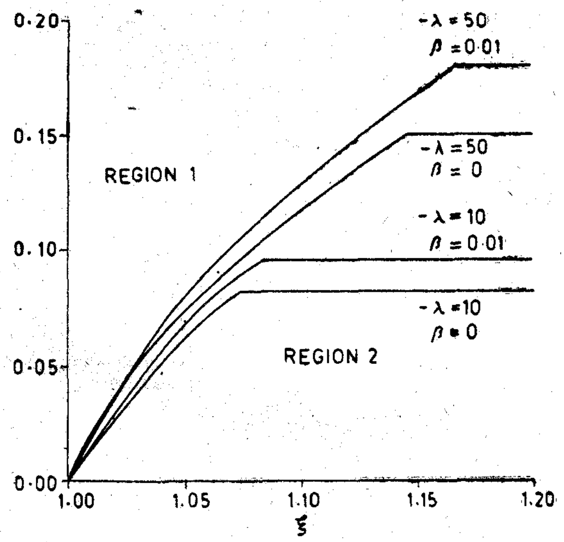

Fig. 4—Characteristics in $\xi - t$ plane (4th degree profile).

against axial distance x. Fig. 3 shows boundary layer thickness against time t as given by the equation (14). The characteristic curves in the ξ—t plane have been drawn in Fig. 4. Each characteristic curve divides the quadrant into two regions marked region 1 and region 2. The region 1 is the range of influence of the initial curve $\xi=1$ (the t-axis for which $t\geqslant 0$). The solution of equation (13) satisfying the boundary condition $\delta=0$ at $\xi=1$ for $t\geqslant 0$ is appropriate for this region. On the other hand, the solution of equation (14) satisfying the boundary condition $\delta=0$ at $t=0$ for $\xi\geqslant 1$ is appropriate in the region 2, which is the range of influence of the initial curve $t=0$ (the ξ— axis for which $\xi\geqslant 1$). Like the fluid injection across the wall of the tube, the tangential velocity slip at the boundaıy also is seen causing an increase in the area of region 2 in Fig. 4. This signifies that the boundary layer thickness is a function of time for a longer period. i.e. the start-up time is increased by both these factors. The estimate of start-up time made by neglecting the velocity slip is less than the time obtained by taking the slip-grouping parameter as 0.01 by about 15%. But these two factors have opposite influence on the entry length by diffusing the retarded fluid in the boundary layer towards the axis.

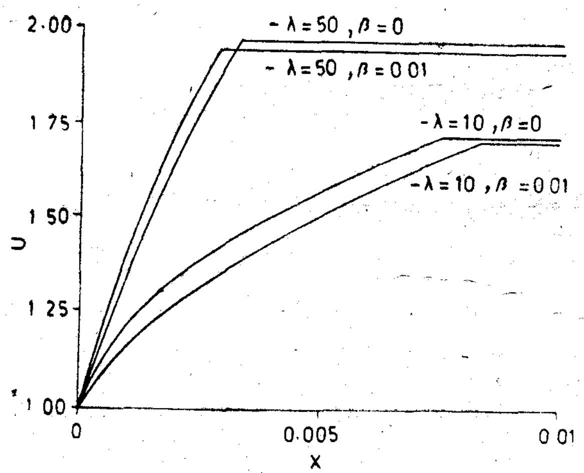

Fig. 5—Core velocity against axial distance (4th degree profile).

The tangential velocity slip, on the other hand reduces the skin friction at the tube wall, thus weakening the cause of boundary layer growth. The estimate of the entry length made on the assumption of no slip at the tube wall is less than the actual entry length in the face of velocity slip. The opposite effect of fluid injection and velocity slip is further illustrated by Fig. 5 showing the core velocity against the axial distance. The injection of fluid across the walls leads to greater quantity of fluid moving near the axis, causing a larger core velocity all through the length of the tube including in its inlet length. On the other hand, by reducing the skin friction at the tube wall, the velocity slip permits larger flow in the region nearer to the wall so that smaller amount of fluid is left to move in the core region. The core velocity is, the refore, smaller in the inlet length of the tube. At smaller value of the injection parameter, the ultimate magnitude of the core velocity obtained on the hypothesis of tangential velocity slip is also less than that obtained on the assumption of no slip, although the difference is not appreciable. But for the larger value of the injection parameter, the ultimate magnitude of core velocity in the length of the tube beyond the inlet region is larger when the velocity slip is taken into account than it is on the assumption of no slip.

A comparison with the earlier study on channel flow shows that the effect of tangential velocity slip is more marked in case of fluid flowing through a tube than it is in case of fluid flowing through a channel. The fluid flowing in a channel bounded by two infinite parallel plates is affected by the boundaries on two sides only. But the fluid flowing in a tube is affected by the condition of tangential velocity slip at the boundary bounding it all round.

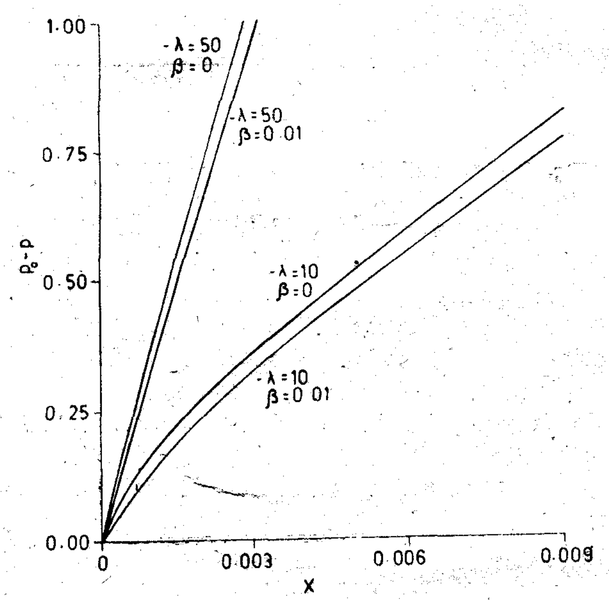

Fig. 6—Pressure drop against axial distance (4th degree profile).

PRESSURE DROP

The pressure gradient in the axial direction is obtained from the equation (10) with the help of (8). For the time beyond the start-up, corresponding to the points lying in the region 1 of the characteristic plane, δ is a function of ξ alone. But the equation (13) being an inverse relationship, does not help in expressing the pressure gradient explicitly in terms of the axial variable. Therefore, the pressure drop has to be obtained indirectly by first evaluating it in terms of the boundary layer thickness δ by integrating numerically the equation.

$$dp/d\delta = 2\,(n+1)\,(n+2)\,(2\,\xi - 1)^2\,[P_2/Q - \{2\,F_2/(2n+1) + n(1/\lambda - \beta)\,Q\}\,G_1/P_3]. \quad (15)$$

From the curves drawn in Fig. 6, it is seen that while the increased fluid injection increases the pressure drop along the axis, the tangential velocity slip mellows it a little. This is as expected from the consideration that in the face of tangential velocity slip, a smaller pressure gradient would suffice to overcome the reduced skin friction.

For points lying in the region 2 of the characteristic plane δ is a function of t but is independent of ξ. Therefore, on treating δ as constant with respect to ξ, the pressure drop along the axis of the tube is obtained as

$$p_o - p = 2\,(n+1)\,(n+2)\,\xi\,(\xi - 1)\,[n/\lambda - \beta n + 2\,F_2/\{(2n+1)\,Q\}\,]/Q. \quad (16)$$

This is valid at any moment during the start-up time only.

NUMERICAL RESULTS FOR $n = 6$

The numerical work presented above lends itself to easy comparison with (7) and (8). The results for the case of $\beta=0$ are in agreement with those in reference (7) although the entry length obtained here is less than that obtained in (8). On taking $n=6$, the entry length is found further greatly reduced. The magnitude of the core velocity at any particular distance from the inlet in the shorter entry length is now slightly more than that obtained by taking $n=4$. The core velocity in the fully developed flow is now much smaller. A comparison of the boundary layer thickness δ and the core velocity U at some points in the entry length obtained by taking $n=6$ and $n=4$ in the two cases of $\lambda=10$ and $\lambda=50$ is made in Tables 1 and 2. But it appears that these features are consequences primarily of the geometrical shape of the sixth degree parabolic profile, which is quite steep near the boundaries $y= \pm 1$ and much flattened in the middle. Its steep gradient near the boundaries leads to smaller values of ξ corresponding to $\delta=1$. The bigger core region

TABLE 1

COMPARATIVE VALUES OF δ AND U AGAINST x for $\lambda = 10$

$\beta = 0.01$				$\beta = 0$			
x	n	δ	U	x	n	δ	U
0.0011	6	0.60	1.1947	0.0010	6	0.65	1.2088
	4	0.39	1.1775		4	0.43	1.2236
0.0020	6	0.82	1.2953	0.0017	6	0.83	1.3110
	4	0.53	1.2698		4	0.55	1.2927
0.0029	6	1.00	1.3845	0.0026	6	1.00	1.4029
	4	0.64	1.3502		4	0.66	1.3800

TABLE 2

COMPARATIVE VALUES OF δ AND U AGAINST x FOR $\lambda = 50$

$\beta = 0.01$				$\beta = 0$			
x	n	δ	U	x	n	δ	U
0.0005	6	0.60	1.2262	0.0004	6	0.63	1.2440
	4	0.40	1.2156		4	0.43	1.2388
0.0008	6	0.82	1.3456	0.0006	6	0.78	1.3265
	4	0.53	1.3172		4	0.54	1.3143
0.0011	6	1.00	1.4542	0.0010	6	1.00	1.4675
	4	0.63	1.4127		4	0.68	1.4470

corresponding to the flattened portion of the velocity profile calls for a smaller core velocity to, cause the same flux across any given cross section. Since the sixth degree parabola does not appear to be a good approximation to the fully developed velocity profile even in the case of mass transfer across the tube wall, its discussion is not pursued in greater detail here. However, such a profile does not seem to be relevant in a short distance in which the plug flow near the inlet changes into the parabolic flow[15]. Therefore it may be examined whether an estimate of the entry length and start-up time can be made by taking $n=6$ for $0<\delta<c$ and $n=4$ for $c<\delta<1$, c being some empirically chosen number lying between 0 and 1, provided that this does not give rise to any analytic discontinuity.

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