FLIGHT-PATH CHARACTERISTICS FOR FEW RE-ENTRY TRAJECTORIES

S. K. GURTU*

Terminal Ballistics Research Laboratory, Chandigarh

(*Received* 1 *April* 1971 ; *revised* 23 *August* 1971)

Approximate analytical solutions on velocity, range, deceleration, maximum deceleration, distance and time of flight, etc. are obtained for object entry in a planetary atmosphere when (i) rate of change of velocity is proportional to the nth power of the velocity, and (ii) rate of change of altitude is proportional to the pth power of the velocity. Loh's results for constant deceleration flight and constant rate of change of descent flight are obtained as particular cases. Finally, trajectory characteristics are obtained for constant rate of change of range flight.

NOTATIONS

A = reference area for lift and drag expressions (sq. ft.)

C_D = drag coefficient

D = drag (lb.)

g = acceleration due to gravity (ft./sec^2)

L = lift (lb.)

m = mass (slugs)

R = range (ft.)

R_0 = radius of earth (ft.)

s = distance along flight-path (ft.)

V = velocity (ft./sec.)

θ = angle of entry

β = atmospheric density coefficient such that $\rho = \rho_0 e^{-\beta y}$ where ρ_0 is atmospheric density at earth's surface and y = altitude.

The problem of vehicle entry into planetary atmosphere is gradually receiving increasing attention. Gazley[1], Allen[2], Eggers[3] and Chappl[4] have considered the case of ballistic-type entry without lift at sufficiently large angles of inclination where both the gravity force and centrifugal force have been neglected.

Loh[5,6] has given analytical solutions on the dynamical aspect of various kinds of re-entry trajectories. He has considered the case for constant and for variable lift-drag ratios. His approach is more general and the results of earlier investigators[1-4] come out as particular cases. The present paper is essentially aimed at generalizing Loh's results for constant deceleration flight and constant rate of change of descent flight and also dealing with a new type of re-entry trajectory. The results for all the above three cases are for variable L/D ratios.

Rate of Change of Velocity Varies as the nth power of Velocity

In a non-rotating two dimensional inertial coordinate system with its origin at the centre of the earth or planet the equations of motion are[5]

$$\frac{dV^2}{ds} + \frac{2g}{L/D}\left[\left(1 - \frac{V^2}{gR_0}\right)\cos\theta - \frac{V^2}{g}\left(\frac{d\theta}{ds}\right)\right] = 0 \tag{1}$$

$$-\left(\frac{C_D A}{m}\right)\left(\frac{1}{2}\rho V\right) = \frac{dV}{dt} = \frac{1}{2}\frac{dV^2}{ds} \tag{2}$$

*Present address :— Defence Science Laboratory, Metcalfe House, Delhi-6.

For high speed re-entry the gravity component in the drag direction is usually small in comparison with drag itself; consequently, it has been neglected in (2).

The trajectory characteristics at

$$\frac{dV}{dt} = K_2 V^n \tag{3}$$

are now worked out

Density:—From (2) and (3) the expression for density is

$$\rho = - \frac{2m K_2}{C_D A V^{2-n}} \tag{4}$$

Altitude:—From the altitude-density relation and equation (4)

$$y = - \frac{1}{\beta} ln \left[- \frac{2m K_2}{\rho_0 C_D A V^{2-n}} \right] \tag{5}$$

Time of flight:—Integrating (3),

$$t = - \frac{1}{K_2 (1-n)} \left[V_i^{1-n} - V^{1-n} \right] \tag{6}$$

where subscript i, here or elsewhere, denotes initial values.

Distance along flight-path:—The distance along flight-path using (6) is

$$s = - \frac{1}{(2-n) K_2} \left(V_i^{2-n} - V^{2-n} \right) \tag{7}$$

Velocity:—The expression for velocity in terms of altitude, is obtained from (2), (3) and the density-altitude relation

$$V = \left(- \frac{2 m K_2}{\rho_0 C_D A} \right)^{1/(2-n)} e^{-\beta y/(2-n)} \tag{8}$$

Angle of inclination:—From relation $\sin \theta = - dy/ds$ and (5) and (7), the entry angle is given by

$$\sin \theta = - \frac{K_2 (2-n)}{\beta V^{2-n}} \tag{9}$$

Lift-drag ratio:—From (1), (7) and (9) on simplification,

$$\frac{L}{D} = \frac{(2-n)^2 K_2}{\beta} \left[V^{4-2n} - \left\{ \frac{K_2 (2-n)}{\beta} \right\}^2 \right]^{-1/2} -$$
$$- \frac{g}{K_2} \left[V^{4-2n} - \left\{ - \frac{K_2 (2-n)}{\beta} \right\}^2 \right]^{1/2} Z \tag{10}$$

where

$$Z = \frac{1}{V^2} - \frac{1}{g R_0} \tag{11}$$

Range:— In the relation $dR = \cos \theta \, ds$, substituting value of $\cos \theta$ from (9) and ds from (7) on integrating and simplifying

$$R = \frac{1}{\beta} \left[\left(\left\{ \frac{\beta V^{2-n}}{(2-n) K_2} \right\}^2 - 1 \right)^{\frac{1}{2}} - \left(\left\{ \frac{\beta V_i^{2-n}}{(2-n) K_2} \right\}^2 - 1 \right)^{\frac{1}{2}} \right] -$$
$$- \frac{1}{\beta} \left[\sec^{-1} \frac{\beta V^{2-n}}{(2-n) K_2} - \sec^{-1} \frac{\beta V_i^{2-n}}{(2-n) K_2} \right] \tag{12}$$

It may perhaps be mentioned here that the expression for range as given by Loh[5] is erroneous. The actual expression can be obtained by putting $n = 0$ in above equation.

When the index n vanishes, equations (4) to (10) give flight-path characteristics for constant deceleration flight[5].

Rate of Change of Altitude varies as the pth Power of the Velocity

The trajectory characteristics are now analysed for entry at

$$\frac{dy}{dt} = K_3 V^p \tag{13}$$

Angle of inclination :—From the relation $\sin\theta = -\,dy/ds$ and above relation.

$$\sin\theta = -\left(\frac{C_{19}\, C_D\, A}{2\,\beta\, m}\right) V^{-1+p} \tag{14}$$

where

$$C_{19} = 2\, m\, \beta\, K_3 / C_D\, A \tag{15}$$

Density :—Using (2) and (13), the relation $d\rho = -\,\beta\,\rho\, dy$ and integrating

$$\rho = C_{18} - \frac{C_{19}}{(1-p)}\, V^{-1+p} \tag{16}$$

where

$$C_{18} = \rho_i + \frac{2\, m\, \beta\, K_3}{C_D\, A\,(1-p)\, V_i^{1-p}} \tag{17}$$

Flight-path distance :—Substituting (16) into (2), integrating, and simplifying

$$s = \left[\frac{2\, m}{C_{18}\, C_D\, A\,(1-p)}\right] ln\, \frac{(1-p)\, C_{18}\, V_i^{1-p} - C_{19}}{(1-p)\, C_{18}\, V^{1-p} - C_{19}} \tag{18}$$

Range :—In the relation $dR = \cos\theta\, ds$ substituting value of $\cos\theta$ from (14) and ds from equation (18) and integrating

$$R = \frac{2\, m\,(1-p)}{C_D\, A} \int_{V_i}^{V} \left[V^{2-2p} - \left(\frac{C_{19}\, C_D\, A}{2\,\beta\, m}\right)^2\right]^{\frac{1}{2}} \cdot$$

$$\cdot \left[C_{19}\, V - C_{18}\, V^{2-p}\,(1-p)\right]^{-1} dV \tag{19}$$

The value of R can be evaluated by numerical integration.

Altitude :—Using relation $\rho = \rho_0\, e^{-\beta y}$ and (16) on simplification

$$y = -\frac{1}{\beta}\, ln\left[e^{-\beta y_i} + \left\{\frac{C_{19}}{(1-p)\,\rho_0}\right\}\left\{V_i^{-1+p} - V^{-1+p}\right\}\right] \tag{20}$$

Velocity :—The expression for velocity, as a function of altitude, follows from above

$$V^{-1+p} = V_i^{-1+p} - \left\{\frac{(1-p)\rho_o}{C_{19}}\right\}\left\{e^{-\beta y} - e^{-\beta y_i}\right\} \tag{21}$$

Deceleration and Maximum Deceleration :—The deceleration expression follows from (2) and (16) as

$$\frac{dV}{dt} = -\left(\frac{C_D\, A}{2\, m}\right)\left[C_{18}\, V^2 - \frac{C_{19}}{(1-p)}\, V^{1+p}\right] \tag{22}$$

and the expression for maximum deceleration is

$$\left(\frac{dV}{dt}\right)_{max} = \left(\frac{C_D A}{2\,m}\right) \frac{C_{19}^{2/(1-p)}}{2^{2/(1-p)}\, C_{18}^{(1+p)/(1-p)}} \left(\frac{1+p}{1-p}\right)^{(1+p)/(1-p)} \qquad (23)$$

where

$$V_{(dV/dt)_{max}} = \left[\left(\frac{C_{19}}{2\,C_{18}}\right)\left(\frac{1+p}{1-p}\right)\right]^{1/(1-p)} \qquad (24)$$

Time of flight:—The expression for flight-time using (18) is

$$t = \int_{V_i}^{V} \frac{ds}{V} = -\left(\frac{2\,m}{C_D A}\right)\int_{V_i}^{V} \frac{(1-p)\,dV}{(1-p)\,C_{18}\,V^2 - C_{19}\,V^{1+p}} \qquad (25)$$

which can be solved by numerical integration.

Lift-drag ratio:—Using equations (1), (14), (18), (22) and simplifying the lift-drag ratio is

$$\frac{L}{D} = (1-p)\left(\frac{2\,g\,m}{C_D A}\right)\frac{[\,V^{2-2p} - K_3^2\,]^{1/2}\,Z}{[\,C_{18}\,(1-p)\,V^{1+p} - C_{19}\,]} + (1-p)\,K_3\,[\,V^{2-2p} - K_3^2\,]^{-1/2} \qquad (26)$$

As before it can be easily seen that if the exponent p is 0, equations (14) to (26) yield results for constant rate of descent flight[5].

Constant Rate of Change of Range Flight

It is now proposed to analyse a different type of re-entry trajectory for which

$$\frac{dR}{dt} = K_1 \qquad (27)$$

Angle of inclination:—The expression for angle of inclination from above is

$$\sin\theta = (V^2 - K_1^2)^{\frac{1}{2}}\,V^{-1} \qquad (28)$$

Density:—Using relations (2), (28), the relations $d\rho = -\beta\,\rho\,dy$ and $\sin\theta = -dy/ds$ and simplifying

$$\rho = C_{24} - C_{23}\left[\,ln\left\{\frac{V + (V^2 - K_1^2)^{\frac{1}{2}}}{K_1}\right\} - \frac{(V^2 - K_1^2)^{\frac{1}{2}}}{V}\,\right] \qquad (29)$$

where

$$C_{24} = \rho_i + C_{23}\left[\,ln\left\{\frac{V_i + (V_i^2 - K_1^2)^{\frac{1}{2}}}{K_1}\right\} - \frac{(V_i^2 - K_1^2)^{\frac{1}{2}}}{V_i}\,\right] \qquad (30)$$

and

$$C_{23} = 2\,\beta\,m/C_D A \qquad (31)$$

Altitude:—Using the altitude-density relation, (29) and simplifying, the altitude in terms of V is

$$y = \frac{1}{\beta} \ln \left[e^{-\beta y_i} - \frac{C_{23}}{\rho_0} \ln \left\{ \frac{V + (V^2 - K_1^2)^{\frac{1}{2}}}{V_i + (V_i^2 - K_1^2)^{\frac{1}{2}}} \right\} + \right.$$

$$\left. + \frac{C_{23}}{\rho_0} \ln \left\{ \frac{(V^2 - K_1^2)^{\frac{1}{2}}}{V} - \frac{(V_i^2 - K_1^2)^{\frac{1}{2}}}{V_i} \right\} \right] \tag{32}$$

Deceleration:—From (2) and (29) the expression for deceleration is

$$\frac{dV}{dt} = \left(- \frac{C_D A}{2m} \right) \left[C_{24} - C_{23} \ln \left\{ \frac{V + (V^2 - K_1^2)^{\frac{1}{2}}}{K_1} \right\} + C_{23} \frac{(V^2 - K_1^2)^{\frac{1}{2}}}{V} \right] V^2 \tag{33}$$

For maximum deceleration

$$\frac{d}{dV} \left(\frac{dV}{dt} \right) = 0$$

which gives the value of V in implicit form as

$$\frac{2C_{24}}{C_{23}} = 2 \ln \left\{ \frac{V}{K_1} + \left[\left(\frac{V}{K_1} \right)^2 - 1 \right]^{\frac{1}{2}} \right\} - \left[1 - \left(\frac{K_1}{V} \right)^2 \right]^{\frac{1}{2}} \tag{34}$$

This value of V when substituted in (33) gives maximum deceleration.

Flight-path distance:—This follows from (2) and (29) on integration

$$s = \left(- \frac{2m}{C_D A} \right) \int_{V_i}^{V} \frac{dV}{V f(V)} \tag{35}$$

where

$$\rho = f(V) = C_{24} - C_{23} \ln \frac{V + (V^2 - K_1^2)^{\frac{1}{2}}}{K_1} + C_{23} \frac{(V^2 - K_1^2)^{\frac{1}{2}}}{V} \tag{36}$$

Range:—From the relation $dR = \cos \theta \, ds$ and (27) and (36), on integration

$$R = \left(\frac{-2mK_1}{C_D A} \right) \int_{V_i}^{V} \frac{dV}{V^2 f(V)} , \tag{37}$$

Time of flight:—The expression for time of flight is

$$t = \int_{V_i}^{V} \frac{ds}{V} = \left(\frac{-2m}{C_D A} \right) \int_{V_i}^{V} \frac{dV}{V^2 f(V)} \tag{38}$$

The last three characteristics, viz flight-path distance, range and time of flight can be evaluated by numerical integration.

CONCLUSIONS

The first entry case at $\frac{dV}{dt} = K_2 V^n$ is particularly interesting. At $n = 0$ it reduces to the constant deceleration flight[5]. It may be mentioned that constant deceleration flight is specially useful for proper entry into Jupiter.

Further, the results for constant rate of average heat input flight[5] can be deduced by putting $n = -1$ and suitably choosing K_2.

The second and third entry cases, however, are more of general interest.

ACKNOWLEDGEMENT

I thank the Director, T. B. R. L., Chandigarh, for providing necessary permission to publish the paper.

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