ASSOCIATED LEGENDRE FUNCTIONS AND COOLING OF A HEATED CYLINDER

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In this paper associated Legendre functions have been employed to obtain solutions of the fundamental differential equation of the cooling of a heated cylinder.

Cases in which heat is produced in solids are becoming increasingly important in technical applications[1]. Space research and nuclear reactors also give rise to different problems of heat transfer.

In this paper we consider the problem of the cooling of infinitely long cylinder of radius a, heated to temperature $u_0 = f(r)$ [r is the distance from the axis] and radiating heat into the surrounding medium at zero temperature. From a mathematical point of view, the problem reduces to solving the equation of heat conduction[2]

$$c\rho.\ \partial u/\partial t = k\,\nabla^2 u\,, \tag{1}$$

subject to the boundary condition

$$(\partial u/\partial r + hu)_{r=a} = 0\,; \tag{2}$$

and the initial condition

$$u\,|_{t=0} = u_0 = f(r)\,, \tag{3}$$

where the object has thermal conductivity k, heat capacity c and density ρ and emissivity λ and $h = \lambda/k$.

Here we shall consider the following values of temperature :

$$f(r) = r^{2\sigma+\lambda-2}\,[a^2 - r^2]^{\mu/2}\,P_\nu^\mu\,(r/a)\,, \tag{4}$$

and

$$f(r) = r^{2\sigma+\lambda-2}\,[a^2 - r^2]^{-\mu/2}\,P_\nu^\mu\,(r/a) \tag{5}$$

The following formulae, recently given by Bajpai[3], are required in the proof.

$$\int_0^a r^{2\sigma+\lambda-1}\,[a^2 - r^2]^{\mu/2}\,P_\nu^\mu\,(r/a)\,J_0\,(x_m\,r/a)\,dr$$

$$= \frac{a^{2\sigma+\lambda+\mu}\,(-1)^\mu\,\Gamma(1+\mu+\nu)\,\Gamma\left(\dfrac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\dfrac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\,\Gamma\left(\dfrac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\dfrac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}\,.$$

$$\cdot\ {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\,; \\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\,;\end{matrix}\ -x_m^2/4\right], \tag{6}$$

where

$$\mathrm{Re}\,(\lambda + 2\sigma) > 0,\ \mu = 0, 1, \ldots$$

and

$$\int_0^a r^{2\sigma+\lambda-1}[a^2-r^2]^{-\mu/2} P_\nu^\mu(r/a)\, J_0(x_m r/a)\, dr$$

$$= \frac{a^{2\sigma+\lambda-\mu}\, \Gamma\left(\frac{1+2\sigma+\lambda}{2}\right) \Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right) \Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)} \cdot$$

$$\cdot\ {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\,; \\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2, (2+2\sigma+\lambda-\mu+\nu)/2\,;\end{matrix} -x_m^2/4\right], \tag{7}$$

where

$\mathrm{Re}\,(\lambda+2\sigma) > 0\ ,\ \mu < 1.$

SOLUTION OF THE PROBLEM

The solutions of (1) to be obtained are

$$u(r,t) = \frac{2a^{2\sigma+\lambda+\mu-2}\,(-1)^\mu\, \Gamma(1+\mu+\nu)\ \Gamma\left(\frac{1+2\sigma+\lambda}{2}\right) \Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\ \Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)} \cdot$$

$$\cdot \sum_{n=1}^{\infty} {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2\ ,(2\sigma+\lambda)/2\,; \\ 1,\ \frac{1+2\sigma+\lambda+\mu-\nu}{2}\ ,\ \frac{2+2\sigma+\lambda+\mu+\nu}{2}\ ;\end{matrix} \quad -x_n^2/4\right] \cdot$$

$$\cdot\ \frac{J_0(x_n r/a)\exp(-x_n^2 t/a^2 b)}{J_0^2(x_n)+J_1^2(x_n)}\ , \tag{8}$$

where

$\mathrm{Re}\,(\lambda+2\sigma) > 0,\quad \mu = 0,\ 1,\ \ldots$

$$u(r,t) = \frac{2a^{2\sigma+\lambda-\mu-2}\ \Gamma\left(\frac{1+2\sigma+\lambda}{2}\right) \Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right) \Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)} \cdot$$

$$\cdot \sum_{n=1}^{\infty} {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2\ ,\ (2\sigma+\lambda)/2\,; \\ 1,\ \frac{1+2\sigma+\lambda-\mu-\nu}{2}\ ,\ \frac{2+2\sigma+\lambda-\mu+\nu}{2}\ ,\end{matrix} \quad -x_n^2/4\right] \cdot$$

$$\cdot\ \frac{J_0(x_n r/a)\exp(-x_n^2 t/a^2 b)}{J_0^2(x_n)+J_1^2(x_n)}\ , \tag{9}$$

where

$\mathrm{Re}\,(\lambda+2\sigma) > 0\ ,\ \mu < 1\ .$

Proof. The solution of (1) is[2]

$$u(r,t) = \sum_{n=1}^{\infty} M_n J_0(x) \exp(-x_n^2 t/a^2 b), \quad (b = c\rho/k) \tag{10}$$

where because of the initial condition (3), the coefficients M_n must be chosen to satisfy the relation

$$f(r) = \sum_{n=1}^{\infty} M_n J_0(x_n r/a), \quad 0 \leqslant r < a. \tag{11}$$

By virtue of (4), we have

$$f(r) = r^{2\sigma+\lambda-2}[a^2 - r^2]^{\mu/2} P_\nu^\mu(r/a)$$

$$= \sum_{n=1}^{\infty} M_n J_0(x_n r/a) \tag{12}$$

Multiplying (12) by $r J_0(x_m r/a)$ and integrating with respect to r from 0 to a, we get

$$\int_0^a r^{2\sigma+\lambda-1}[a^2 - r^2]^{\mu/2} P_\nu^\mu(r/a) J_0(x_m r/a)\, dr = \sum_{n=1}^{\infty} M_n \int_0^a r J_0(x_n r/a) J_0(x_m r/a)\, dr \tag{13}$$

Now using (6) and the orthogonality property of Bessel functions[2], viz.,

$$\int_0^a r J_\nu(x_{\nu m} r/a) J_\nu(x_{\nu n} r/a)\, dr = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{a^2}{2}[J'^2_\nu(x_n) + (1 - \nu/x_{\nu n}^2) J_\nu^2(x_{\nu n})], & \text{if } m = n \end{cases},$$

we obtain

$$M_n = \frac{2a^{2\sigma+\lambda+\mu-2}(-1)^\mu \Gamma(1+\mu+\nu)\, \Gamma\left(\frac{1+2\sigma+\lambda}{2}\right) \Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\, \Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)} .$$

$$\cdot \frac{1}{[J'^2_0/(x_m) + J_0^2/(x_m)]^2} .$$

$$\cdot\, {}_2F_3 \left[\begin{matrix} (1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2; \\ 1,\ \dfrac{1+2\sigma+\lambda+\mu-\nu}{2},\ \dfrac{2+2\sigma+\lambda+\mu+\nu}{2}; \end{matrix} \quad -x_m^2/4 \right] \tag{14}$$

Now with the help of the relation $J_0'(z) = -J_1(z)$, the solution (8) follows from (10) and (14) immediately.

The solution (9) can be obtained similarly by virtue of the relation (5) and (7).

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