TRIPLE INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS

B. M. Singh

Samart Ashok Technological Institute, Vidisha (M. P.)

(*Received 12 April 1971; revised 26 August 1971*)

An exact solution of triple integral equations involving inverse Mellin transforms has been obtained by finite Hilbert transform technique in this paper. The advantage of the above mentioned technique is that the solution obtained is simpler than that given by Srivastav & Parihar. Finally the application of triple integral equations to two dimensional electrostatic problem has also been discussed.

This paper is devoted to the study of the integral equations

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi(s)\,\rho^{-s}\,ds = f_1(\rho), \quad 0<\rho<a,$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(s)\tan\alpha s\,\rho^{-s}\,ds = f_2(\rho), \quad a<\rho<1,$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi(s)\,\rho^{-s}\,ds = f_3(\rho), \quad \rho>1,$$

where $\psi(s)$ is unknown and $f_1(\rho)$, $f_2(\rho)$ and $f_3(\rho)$ are prescribed. Initially Srivastav & Parihar[1] solved dual and triple integral equations involving inverse Mellin transforms in a closed form. They reduced triple integral equations to dual series equations, whose solution is well-known. Later on Erdelyi[2] presented the solution of dual integral equations with the help of fractional integrals which leads to the solution already obtained by Srivastav & Parihar[1]. The method adopted here is that of finite Hilbert transform technique discussed by Srivastava, & Lowengrub[3], for solving triple integral equations. It will serve to get an straight and simplified solution. In the final section an application has been made to electrostatic problem. The analysis given here is purely formal and no attempt is made to justify the various limiting process.

RESULTS

Some results, which are useful later on are given here.

From Mellin's inversion theorem and integral relations given[4], it follows that for $c>0$

$$\left.\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^{-1}B\left(\frac{1}{2},\frac{s}{h}\right)\rho^{-s}\,ds &= (1-\rho^h)^{-\frac{1}{2}} \\ &\qquad \text{for } 0<\rho<1, \\ &= 0 \\ &\qquad \text{for } \rho>1,\end{aligned}\right\} \tag{1}$$

and that for $c<\dfrac{h}{2}$,

$$\left.\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^{-1}B\left(\frac{1}{2},\frac{1}{2}-\frac{s}{h}\right)\rho^{-s}\,ds &= 0, \\ &\qquad \text{for } 0<\rho<1, \\ &= (\rho^h-1)^{-\frac{1}{2}} \\ &\qquad \text{for } \rho>1.\end{aligned}\right\} \tag{2}$$

Trivial change of variables can easily be found from Gradshteyn & Ryzhik[5] and from[3]

$$\int_a^b \frac{t^{h/2-1}\,dt}{(t^h-a^h)^{1/2}\,(b^h-t^h)^{1/2}} = \frac{2F\left(\frac{\pi}{2}, q\right)}{h\,b^{h/2}}, \tag{3}$$

$$\left.\begin{aligned} \int_a^b \frac{\frac{h}{2}\rho^{h/2-1}\log\left|\frac{(t^{h/2}+\rho^{h/2})}{(t^{h/2}-\rho^{h/2})}\right| d\rho}{[(\rho^h-a^h)\,(b^h-\rho^h)]^{1/2}} &= \frac{\pi}{b^{h/2}}\,F\left[\sin^{-1}\frac{t^{h/2}}{a^{h/2}}, \frac{a^{h/2}}{b^{h/2}}\right], 0<t\leqslant a, \\ &= \frac{\pi}{b^{h/2}}\,K\left(\frac{a^{h/2}}{b^{h/2}}\right), a<t<b, \\ &= \frac{\pi}{b^{h/2}}\,F\left(\sin^{-1}\frac{b^{h/2}}{t^{h/2}}, \frac{a^{h/2}}{b^{h/2}}\right), t>b, \end{aligned}\right\} \tag{4}$$

where $q=\frac{(b^h-a^h)^{\frac{1}{2}}}{b^h}$ and F is elliptic integral of first kind. K denotes the complete elliptic integral. Tricomi[6] discussed the theorem for finite Hilbert transform, but here we shall use the modified Hilbert transform theorem.

MODIFIED HILBERT TRANSFORM THEOREM

If $p \in L_2\,(a, b)$, then the equation

$$F_0\,[g\,(t^h)] = \frac{1}{\pi}\int_a^b \frac{h\,t^{h-1}\,g\,(t^h)\,dt}{t^h-y^h} = p\,(y),\; y \in (a, b), \tag{5}$$

has the solution

$$\overset{-1}{F_y}\,[p\,(y)] = g\,(t^h) = -\frac{1}{\pi}\left(\frac{t^h-a^h}{b^h-t^h}\right)^{\frac{1}{2}}\int_a^b\left(\frac{b^h-y^h}{y^h-a^h}\right)^{\frac{1}{2}}\frac{p\,(y)\,h\,y^{h-1}\,dy}{y^h-t^h}$$

$$+\frac{C}{(t^h-a^h)^{\frac{1}{2}}\,(b^h-t^h)^{\frac{1}{2}}}, \tag{6}$$

where C is an arbitrary constant and the first term belongs to the class $L_2\,(a,\ b)$ (F_y is called the finite Hilbert transform.)

SOLUTION OF TRIPLE INTEGRAL EQUATIONS

Let us consider the triple integral equations

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi\,(s)\,\rho^{-s}\,ds = f_1\,(\rho),\; 0<\rho<a, \tag{7}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi\,(s)\tan as\,\rho^{-s}\,ds = f_2\,(\rho),\; a<\rho<1, \tag{8}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi\,(s)\,\rho^{-s}\,ds = f_3\,(\rho),\; \rho>1. \tag{9}$$

Assuming

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s\,\psi(s)\,\rho^{-s}\,ds = \rho^{h/2}\,g(\rho^h), \quad a<\rho<1, \tag{10}$$

Using Mellin's theorem and (7), (9) and (10), we obtain :

$$\psi(s) = \frac{1}{s}\left[\int_0^a f_1(t)\,t^{s-1}\,dt + \int_a^1 t^{s+h/2-1}\,g(t^h)\,dt + \int_1^\infty f_3(t)\,s^{s-1}\,dt\right]. \tag{11}$$

Substituting (11) in (8), we have after interchanging the order of integrations

$$\int_a^1 t^{h/2-1}\,g(t^h)\,dt.\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\rho^{-s}\,t^s\tan as\,ds}{s} = H(\rho), \quad a<\rho<1, \tag{12}$$

where

$$H(\rho) = f_2(\rho) - \int_0^a f_1(t)\,dt\,.\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^{-1+s}\,\rho^{-s}\tan as\ \ ds}{s} -$$

$$-\int_1^\infty f_3(t)\,dt\ .\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^{-1+s}\,\rho^{-s}\tan as\,ds}{s}\ . \tag{13}$$

Using (1) and (2), it can easily be proved that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^s\,\rho^{-s}\tan as\,ds}{s} = \frac{h}{\pi}\int_0^{min(\rho,\,t)}\frac{x^{h-1}\,dx}{(\rho^h-x^h)^{1/2}\,(t^h-x^h)^{1/2}}, \tag{14}$$

where $a = \dfrac{\pi}{h}$.

With the help of well-known integral

$$\int_0^{min(x,\,t)}[(x-y)(t-y)]^{-1/2}\,dy = \log\left|\frac{(x+t)}{(x-t)}\right|, \tag{15}$$

(14) can be written as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^s\,\rho^{-s}\tan as\,ds}{s} = \frac{h}{\pi}\int_0^{min(\rho,\,t)}\frac{x^{h-1}\,dx}{(\rho^h-x^h)^{1/2}\,(t^h-x^h)^{1/2}}$$

$$= \frac{1}{\pi}\log\left|\frac{(\rho^{h/2}+t^{h/2})}{(\rho^{h/2}-t^{h/2})}\right|. \tag{16}$$

Using (16), we can now write (12) in the form

$$\frac{1}{\pi}\int_a^1 t^{h/2-1}\,g(t^h)\log\left|\frac{(\rho^{h/2}+t^{h/2})}{(\rho^{h/2}-t^{h/2})}\right|dt = H(\rho), \quad a<\rho<1. \tag{17}$$

Differentiating (17) with respect to ρ, we have

$$\frac{1}{\pi}\int_a^1\frac{h\,t^{h-1}\,g(t^h)\,dt}{(t^h-\rho^h)} = \frac{H'(\rho)}{\rho^{h/2-1}}, \quad a<\rho<1, \tag{18}$$

where $H'(\rho)$ denotes differentiation with respect to ρ. The equation (18) has the same form as (5), using (6), the function $g(t^h)$ is given by

$$g(t^h) = -\frac{h}{\pi}\left(\frac{t^h - a^h}{1 - t^h}\right)^{\frac{1}{2}} \int_a^1 \left(\frac{1-\rho^h}{\rho^h - a^h}\right)^{\frac{1}{2}} \frac{\rho^{h/2}}{(\rho^h - t^h)} H'(\rho)\, d\rho +$$

$$+ \frac{C}{(t^h - a^h)^{\frac{1}{2}}(1 - t^h)^{\frac{1}{2}}}, \qquad a < t < 1. \tag{19}$$

For determining C, multiplying both sides of (17) by

$$[(\rho^h - a^h)(1 - \rho^h)]^{-\frac{1}{2}} \frac{h}{2} \rho^{(h/2-1)}$$

and integrating between the limit $(a, 1)$, we get on using (4)

$$\int_a^1 t^{h/2-1} g(t^h)\, dt = \frac{1}{K(a^{h/2})} \int_a^1 \frac{\frac{h}{2} H(\rho)\rho^{h/2-1}\, d\rho}{(\rho^h - a^h)^{\frac{1}{2}}(1-\rho^h)^{\frac{1}{2}}}. \tag{20}$$

Substituting (19) into (20) and then interchanging the order of integrations, we find that

$$C = \frac{h^2}{4F\left(\frac{\pi}{2}, q\right)K(a^{h/2})} \int_a^1 \frac{H(\rho)\rho^{h/2-1}\, d\rho}{(\rho^h - a^h)^{\frac{1}{2}}(1-\rho^h)^{\frac{1}{2}}} +$$

$$+ \frac{h^2}{2\pi F\left(\frac{\pi}{2}, q\right)} \int_a^1 \left(\frac{t^h - a^h}{1 - t^h}\right)^{\frac{1}{2}} t^{h/2-1}\, dt\,.$$

$$\cdot \int_a^1 \left(\frac{1-\rho^h}{\rho^h - a^h}\right)^{\frac{1}{2}} \frac{\rho^{h/2} H'(\rho)\, d\rho}{\rho^h - t^h}, \tag{21}$$

where $\quad q = (1 - a^h)^{\frac{1}{2}}$.

Knowing $g(t^h)$ with the help of (19) and (21) we can get $\psi(s)$ from (11).

The special case of much practical importance shall be one in which $f_1 = f_3 = 0$, $f_2(\rho) = p_0$, then

$$g(t^h) = \frac{C}{(t^h - a^h)^{1/2}(1 - t^h)^{1/2}}; \tag{22}$$

where

$$C = \frac{hp_0}{2K(a^{h/2})}.$$

TWO DIMENSIONAL ELECTROSTATIC PROBLEMS

To illustrate the use of the solution of triple integral equations we consider the wedge shaped region fangle α, earthed on the face $\theta = \alpha$ and a strip lying in the interval $a < \rho < 1$ charged at a constant otential p_0. The strip is lying on the face $\theta = 0$.

We have the following conditions on the potential function V.

$$\nabla^2 V(\rho, \theta) = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \theta^2} = 0 \tag{23}$$

$$\rho \in (0, \infty),$$
$$0 < \theta < \alpha,$$

$$V(\rho, \alpha) = 0, \ \theta = \alpha, \qquad 0 < \rho < \infty \tag{24}$$

$$V(\rho, 0) = p_0 \text{ (constant)}, \qquad a < \rho < 1 \tag{25}$$

$$\left(\frac{1}{\rho} \frac{\partial V}{\partial \theta}\right)_{\theta=0} = 0, \qquad 0 < \rho < a, \ \rho > 1. \tag{26}$$

The solution of (23) satisfying the condition (24), has the form

$$V(\rho, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A(s) \sin(\alpha - \theta) s \, \rho^{-s} \, ds. \tag{27}$$

(25) and (26) reduce to triple integral equations.

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \, \psi(s) \, \rho^{-s} \, ds = 0, \qquad 0 < \rho < a, \ 1 < \rho \tag{28}$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(s) \tan \alpha s \rho^{-s} \, ds = p_0, \qquad a < \rho < 1 \tag{29}$$

where $\psi(s) = A(s) \cos \alpha s$.

Now we shall find the surface charge density of the strip which is equal to

$$-\frac{1}{4\pi\rho} \left(\frac{\partial V}{\partial \theta}\right)_{\theta=0} = \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \, A(s) \cos \alpha s \ \rho^{-s} \, ds\right] \frac{1}{4\pi\rho},$$

$$a < \rho < 1. \tag{30}$$

Hence

$$\sigma(\rho) = \frac{1}{4\pi\rho} \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \, \psi(s) \, \rho^{-s} \, ds\right]; \qquad a < \rho < 1 \tag{31}$$

We have from (11)

$$\psi(s) = \frac{1}{s} \int_a^1 t^{s + h/2 - 1} \, g(t^h) \, dt. \tag{32}$$

Using the Mellin's theorem from (28) and (31) we get

$$s \, \psi(s) = 4\pi \int_a^1 \sigma(\rho) \, \rho^s \, d\rho. \tag{33}$$

From (32) and (33) we have

$$\sigma(t) = \frac{t^{h/2-1} g(t^h)}{4\pi}, \quad a < t < 1. \tag{34}$$

Hence

$$\sigma(t) = \frac{hp_0 t^{h/2-1}}{8\pi K(a^{h/2}) \left[(t^h - a^h)^{\frac{1}{2}} (1 - t^h)^{\frac{1}{2}} \right]}, \; a < t < 1 \tag{35}$$

where $\alpha = \pi/h$.

ACKNOWLEDGEMENTS

I thank Dr. P. M. Gupta, Professor and Head of the Department of Mathematics, Samrat Ashok Technological Institute, Vidisha, M. P. for helping me in preparation of this paper and to Dr. R. D. Agarwal for helpful suggestions.

REFERENCES

1. Srivastav, R. P. & Parihar, K. S., *SIAM J. Appl. Math.*, **16** (1968).
2. Erdelyi, A., *SIAM J. Appl. Math.*, **16** (1968).
3. Srivastava, K. N. & Lowengrub, M., *Proc. R. Soc.*, Edinb **18**, 1970.
4. Erdelyi, A. W., Magnus, F., Oberhettinger & Tricomi, F. G., "Tables of integral transforms", Vol. 1, (McGraw Hill, New York) 1954, 311.
5. Gradshteyn, I. S., & Ryzhik, I. M., "Tables of Integral, Series and Products", (Academic Press, New York), 1965.
6. Tricomi, F. G., "Integral Equations," (Interscience Publishers Inc., New York), 1967.