SOME THEOREMS ON GENERALIZED BASIC HYPERGEOMETRIC SERIES

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In an earlier paper the author has established two theorems on generalized hypergeometric functions. In each theorem a numerator differs from a denominator by a positive integer. These theorems were further used to prove some theorems on the sums of Kampé de Fériet function. Here, we have established the theorems which are the basic analogues of the theorems proved in the earlier paper.

The author[1] has established two theorems on the sums of generalized hypergeometric functions. In the first theorem, a ${}_{i+1}F_{j+1}$, in which a numerator parameter exceeds a denominator one by a positive integer, say m, has been expressed as the sum of $m+1$ ${}_iF_j$'s. In the second theorem a ${}_{i+1}F_{j+1}$, in which a denominator parameter exceeds a numerator parameter by a positive integer, say n, has been expressed as the sum of $n+1$ ${}_{i+1}F_{j+1}$'s. The two theorems were further employed to establish some theorems on the sums of hypergeometric functions of two variables. The aim of the present paper is to give the basic analogues of the theorems established earlier.

Assuming $|q| < 1$, and n, a positive integer, let

$$[a]_n = (1-a)(1-aq)\ldots(1-aq^{n-1});\ [a]_0=1 \tag{1}$$

$$[a]_{-n} = (-1)^n\, q^{\Sigma n}\, a^{-n}\, [q/a]_n, \tag{2}$$

where

$$\Sigma n = 1+2+\ldots+n.$$

The generalized basic hypergeometric functions of one and two variables have been defined as

$$ {}_i\Phi_j(a_i;\ b_j;\ x) = \sum_{r=0}^{\infty} \frac{\Pi\,[a_i]_r\, x^r}{\Pi\,[b_j]_r\,[q]_r},\ |x|<1 \tag{3}$$

$$\Phi\begin{matrix} g:h;H \\ \\ l:k;K\end{matrix}\left[\begin{matrix} a_g : b_h; B_H; \\ & x, y \\ \alpha_l : c_k; C_K; \end{matrix}\right] = \sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{\Pi[a_g]_{p+r}\ \Pi[b_h]_p\ \Pi[B_H]_r\ x^p y^r}{\Pi[\alpha_l]_{p+r}\ \Pi[c_k]_p\ \Pi[C_K]_r\ [q]_p\ [q]_r} \tag{4}$$

where $|x|+|y|<1$. Here and in what follows $\Pi[\mu_i]_s$ stands for the product $[\mu_1]_s\,[\mu_2]_s\ldots[\mu_i]_s$; a_m denotes the set of parameters $a_1, a_2, \ldots, a_m$. The colon (:) and semicolon (;) separate the terms of the type $[a_g]_{p+r}$ and $[b_h]_p$, $[B_H]_r$ on the left of (4).

If n is a positive integer, then by a q-binomial coefficient we mean

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} = \frac{[q]_n}{[q]_k\,[q]_{n-k}},\quad (0 \leqslant k \leqslant n) \tag{5}$$

Clearly

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1 \tag{6}$$

$$\begin{bmatrix} n \\ r \end{bmatrix} + q^{r+1}\begin{bmatrix} n \\ r+1 \end{bmatrix} = \begin{bmatrix} n+1 \\ r+1 \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} q^{n-r} + \begin{bmatrix} n \\ r+1 \end{bmatrix} \tag{7}$$

The author[2] has recently given a contiguous relation

$$\left[\frac{1-a}{a} - \frac{1-bq^{-1}}{bq^{-1}}\right]\Phi(a_i, a;\ b_j, b;\ x) = \frac{1-a}{a}\,\Phi(a_i, aq;\ b_j, b;\ x) - \frac{1-bq^{-1}}{bq^{-1}}\,\Phi(a_i, a;\ b_j, bq^{-1};\ x) \tag{8}$$

Theorems :

Theorems to be established are :

$$ {}_{i+1}\Phi_{j+1}(a_i, aq^n; b_j, a; x) = \sum_{\lambda=0}^{n} \begin{bmatrix} n \\ \lambda \end{bmatrix} \frac{\Pi\,[a_i]_\lambda\ (axq^{\lambda}-1)^\lambda}{\Pi[b_j]_\lambda[a]_\lambda}\ {}_i\Phi_j\,(a_i q^\lambda; b_j q^\lambda\ ;\ x) \tag{9} $$

$$ {}_{i+1}\Phi_{j+1}\,(a_i, aq^{-n}; b_j, a; x) = \sum_{\lambda=0}^{n} \begin{bmatrix} n \\ \lambda \end{bmatrix} \frac{\Pi[a_i]_\lambda\ (-ax)^\lambda\ q^{-n\lambda+\lambda(\lambda-1)/2}}{[b_j]_\lambda\ [a]_\lambda} \cdot $$

$$ \cdot\ {}_{i+1}\Phi_{j+1}\,(a_i\, q^\lambda, a; b_j\, q^\lambda; aq^\lambda\ ; x) \tag{10} $$

Proof :

We shall prove the above theorems by induction on n. We have

$$ {}_{i+1}\Phi_{j+1}\,(a_i, aq; b_j, a; x) = \sum_{r=0}^{\infty} \frac{\Pi[a_i]_r\, x^r}{\Pi[b_j]_r\,[q]_r} \cdot \frac{1-aq^r}{1-a} $$

$$ = \sum_{r=0}^{\infty} \frac{\Pi[a_i]_r\, x^r}{\Pi[b_j]_r\ [q]_r} \left[\, 1 + \frac{a(1-q^r)}{1-a} \right] $$

$$ = {}_i\Phi_j\,(a_i; b_j; x) + \frac{\Pi[a_i]_1\,(ax)}{\Pi[b_j]_1\ [a]_1}\ {}_i\Phi_j\,(a_i\, q; b_j\, q; x) $$

Thus (9) holds for $n=1$. Let us assume that it holds for $n=m$. Further

$$ {}_{i+1}\Phi_{j+1}\,(a_i, aq^{m+1}, b_j, a; x) $$

$$ = \sum_{r=0}^{\infty} \frac{\Pi[a_i]_r\, x^r\, [aq^m]_r}{\Pi[b_j]_r\,[q]_r\,[a]_r} \cdot \frac{1-aq^{m+r}}{1-aq^m} $$

$$ = \sum_{r=0}^{\infty} \frac{\Pi[a_i]_r\, x^r\, [aq^m]_r}{\Pi[b_j]_r\,[q]_r\,[a]_r} \left[\, 1 + \frac{a(1-q^r)}{1-aq^m} \right] $$

$$ = {}_{i+1}\Phi_{j+1}\,(a_i, aq^m; b_j, a; x) + \frac{\Pi[a_i]_1\, x\, a\, q^m}{\Pi[b_j]_1\,[a]_1}\ {}_{i+1}\Phi_{j+1}\,(a_i\, q, aq^{m+1}; b_j\, q, aq; x) $$

Now using the result (9) for $n = m$, this becomes

$$ \sum_{\lambda=0}^{m} \begin{bmatrix} m \\ \lambda \end{bmatrix} \frac{\Pi[a_i]_\lambda\ (axq^\lambda-1)^\lambda}{\Pi[b_j]_\lambda\ [a]_\lambda}\ {}_i\Phi_j\,(a_i\, q^\lambda\ ; b_j\, q^\lambda\ ; x) + $$

$$ + \frac{\Pi[a_i]_1\ \ axq^m}{\Pi[b_j]_1\ [a]_1} \sum_{\lambda=0}^{m} \begin{bmatrix} m \\ \lambda \end{bmatrix} \frac{\Pi[a_i q]_\lambda\ \ (axq)^\lambda}{\Pi[b_j\, q]_\lambda\ \ [aq]_\lambda}\ q^{\lambda(\lambda-1)}\ -\ {}_i\Phi_j\,(a_i q^{\lambda+1}; b_j q^{\lambda+1}; x) $$

$$= \begin{bmatrix} m \\ 0 \end{bmatrix} {}_i\Phi_j(a_i; b_j; x) + \sum_{\lambda=1}^{m} \frac{\Pi[a_i]_\lambda\,(axq^{\lambda-1})^\lambda}{\Pi[b_j]_\lambda [a]_\lambda} \cdot \left[\begin{bmatrix} m \\ \lambda \end{bmatrix} + q^{m-\lambda+1}\begin{bmatrix} m \\ \lambda-1 \end{bmatrix}\right] {}_i\Phi_j(a_i q^\lambda ; b_j q^\lambda ; x) +$$

$$+ \begin{bmatrix} m \\ m \end{bmatrix} \frac{\Pi[a_i]_{m+1}\,(axq^m)^{m+1}}{\Pi[b_j]_{m+1}\,[a]_{m+1}}\, {}_i\Phi_j\,(a_i\, q^{m+1}; b_j\, q^{m+1}; x)$$

The result (9) now follows from (6) and (7).

That formula (10) holds for $n=1$, can easily be proved. Let us assume that it holds for $n=m$. Also (8) yields

$${}_{i+1}\Phi_{j+1}(a_i, aq^{-m-1}; b_j, a; x) - {}_{i+1}\Phi_{j+1}(a_i, aq^{-m}; b_j, a; x)$$

$$= -\frac{1-aq^{-1}}{1-q^m}\,[{}_{i+1}\Phi_{j+1}(a_i, aq^{-m}; b_j, a; x) - {}_{i+1}\Phi_{j+1}(a_i, aq^{-m-1}; b_j, aq^{-1}; x) \tag{11}$$

Now using the assumption that (10) holds for $n=m$, we see that the right hand side of (11), on using the summation notation, becomes

$$-\frac{1-aq^{-1}}{1-q^m}\sum_{\lambda=0}^{m}\sum_{r=0}^{\infty}\begin{bmatrix} m \\ \lambda \end{bmatrix}\frac{\Pi[a_i]_{\lambda+r}\,(-ax)^\lambda\, x^r\, q^{-\lambda m+\lambda(\lambda-1)/2}}{[b_j]_{\lambda+r}\,[q]_r}\left[\frac{[a]_r}{[a]_{\lambda+r}} - \frac{q^{-\lambda}[aq^{-1}]_r}{[aq^{-1}]_{\lambda+r}}\right]$$

$$= -\frac{1-aq^{-1}}{1-q^m}\sum_{\lambda=0}^{m}\sum_{r=0}^{\infty}\begin{bmatrix} m \\ \lambda \end{bmatrix}\frac{\Pi[a_i]_{\lambda+r}(-ax)^\lambda x^r\, q^{-\lambda[m-(\lambda-1)/2]}}{\Pi[b_j]_{\lambda+r}[q]_r}\,(1-q^{-\lambda})\,\frac{[aq^{-1}]_r}{[aq^{-1}]_{\lambda+r+1}}$$

$$= \sum_{\lambda=1}^{m}\sum_{r=0}^{\infty}\begin{bmatrix} m-1 \\ \lambda-1 \end{bmatrix}\frac{\Pi[a_i]_{\lambda+r}(-ax)^\lambda\, x^r q^{-\lambda[m-(\lambda-1)/2]}q^{-\lambda}}{\Pi[b_j]_{\lambda+r}\,[q]_r}\cdot\frac{[a]_r}{[a]_{\lambda+r}}\left(\frac{1-aq^{-1}}{1-aq^{r-1}}\right)$$

$$= \sum_{\lambda=1}^{m}\sum_{r=0}^{\infty}\begin{bmatrix} m-1 \\ \lambda-1 \end{bmatrix}\frac{\Pi[a_i]_{\lambda+r}(-ax)^\lambda x^r q^{-\lambda[m-(\lambda-1)/2]}}{\Pi[b_j]_{\lambda+r}\;[q]_r}\cdot\frac{[a]_r}{[a]_{\lambda+r}}\left(1-\frac{aq^{-1}(1-q^r)}{1-aq^{r-1}}\right)$$

$$= \sum_{\lambda=1}^{m}\begin{bmatrix} m-1 \\ \lambda-1 \end{bmatrix}\frac{\Pi[a_i]_\lambda\,(-ax)^\lambda\, q^{-\lambda[m+1-(\lambda-1)/2]}}{\Pi[b_j]_\lambda\,[a]_\lambda}\,{}_{i+1}\Phi_{j+1}[a_i q^\lambda, a; b_j q^\lambda, aq^\lambda; x] -$$

$$-\sum_{\lambda=2}^{m+1}\sum_{r=1}^{\infty}\begin{bmatrix} m-1 \\ \lambda-2 \end{bmatrix}\frac{\Pi[a_i]_{\lambda+r-1}[a]_{r-1}(-ax)^\lambda x^{r-1} q^{-(\lambda-1)[m+1-(\lambda-2)/2]-1}}{\Pi[b_j]_{\lambda+r-1}[a]_\lambda[q]_{r-1}[aq^\lambda]_{r-1}}$$

Here we have changed λ to $\lambda-1$ in the second expression).

$$= \begin{bmatrix} m-1 \\ 0 \end{bmatrix} \frac{[a_i]_1 (-ax)\, q^{-m-1}}{[b_j]_1\, [a]_1} \; {}_{i+1}\Phi_{j+1} (a_i q, a; b_j q, aq; x) +$$

$$+ \sum_{\lambda=2}^{m} \left[\begin{bmatrix} m-1 \\ \lambda-1 \end{bmatrix} + q^{m-\lambda+1} \begin{bmatrix} m-1 \\ \lambda-2 \end{bmatrix} \right] \frac{\Pi[a_i]_\lambda (-ax)^\lambda q^{-\lambda[m+1-(\lambda-1)/2]}}{\Pi[b_j]_\lambda\, [a]_\lambda} \; {}_{i+1}\Phi_{j+1} (a_i q^\lambda, a; b_j q^\lambda, aq^\lambda; x) +$$

$$+ \begin{bmatrix} m-1 \\ m-1 \end{bmatrix} \frac{\Pi[a_i]_{m+1}\, (-ax)^{m+1}\, q^{-(m+1)(m+2)/2}}{\Pi[b_j]_{m+1}\, [a]_{m+1}} \; {}_{i+1}\Phi_{j+1} (a_i q^{m+1}, a; b_j q^{m+1}, aq^{m+1}; x)$$

Substituting (11), using (10) for $n = m$ again on the left hand side and using the results (6) and (7) we see that the theorem holds for $n=m+1$. This completes the proof of (10).

Remark.—If in a form ${}_i\Phi_j$, k of the numerator parameters exceed k of the denominator parameters by positive integers, say n_t $(t=1, 2, \ldots, k)$, the form ${}_i\Phi_k$ can be expressed as the total sum of $(n_1+1)(n_2+1) \ldots (n_k+1)$ ${}_{i-k}\Phi_{j-k}$.

HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

We shall establish the following theorems on the sums of basic hypergeometric functions of two variables :

$$\Phi^{g:h;H+1}_{l:k;K+1} \begin{bmatrix} a_g : b_h;\, B_H,\, aq^n; \\ \qquad\qquad\qquad x,\, y \\ \alpha_l : c_k;\, C_K,\, a; \end{bmatrix}$$

$$= \sum_{\lambda=0}^{n} \begin{bmatrix} n \\ \lambda \end{bmatrix} \frac{\Pi[a_g]_\lambda\, \Pi[B_H]_\lambda\, (ayq^{\lambda-1})^\lambda}{\Pi[\alpha_l]_\lambda\, \Pi[C_K]_\lambda\, [a]_\lambda} \; \Phi^{g:h;H}_{l;k;K} \begin{bmatrix} a_g q^\lambda;\, b_h;\, B_H q^\lambda; \\ \qquad\qquad x,\, y \\ \alpha_l q^\lambda;\, c_k,\, C_K q^\lambda \end{bmatrix} \tag{12}$$

$$\Phi^{g:h+1;H+1}_{l:k+1;K+1} \begin{bmatrix} a_g : b_h,\, aq^m;\, B_H,\, Aq^n; \\ \qquad\qquad\qquad x,\, y \\ \alpha_l : c_k,\, a;\, C_K,\, A; \end{bmatrix}$$

$$= \sum_{\lambda=0}^{m} \sum_{\mu=0}^{n} \begin{bmatrix} m \\ \lambda \end{bmatrix} \begin{bmatrix} n \\ \mu \end{bmatrix} \frac{\Pi[a_g]_{\lambda+\mu}\, \Pi[b_h]_\lambda\, \Pi[B_H]_\mu\, (ax)^\lambda\, (Ay)^\mu}{\Pi[\alpha_l]_{\lambda+\mu}\, \Pi[c_k]_\lambda\, \Pi[C_K]_\mu\, [a]_\lambda\, [A]_\mu}\, q^{\lambda(\lambda-1)+\mu(\mu-1)} \cdot$$

$$\cdot\, \Phi^{g:h;H}_{l:k;K} \begin{bmatrix} a_g q^{\lambda+\mu} : b_h q^\lambda;\, B_H q^\mu; \\ \qquad\qquad x,\, y \\ \alpha_l q^{\lambda+\mu} : c_k q^\lambda;\, C_K q^\mu; \end{bmatrix} \tag{13}$$

$$\Phi^{g+1:h;H}_{l+1:k;K} \begin{bmatrix} a_g,\, aq^n : b_h;\, B_H; \\ \qquad\qquad x,\, y \\ \alpha_l,\, a : c_k;\, C_K; \end{bmatrix}$$

$$= \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n-\lambda} \begin{bmatrix} n \\ \lambda \end{bmatrix} \begin{bmatrix} n-\lambda \\ \mu \end{bmatrix} \cdot \frac{\Pi[a_g]_{\lambda+\mu}\, \Pi[b_h]_\mu\, \Pi[B_H]_\lambda\, a^{\lambda+\mu}\, y^\lambda\, x^\mu}{\Pi[\alpha_l]_{\lambda+\mu}\, \Pi[c_k]_\mu\, \Pi[C_K]_\lambda\, [a]_{\lambda+\mu}} \cdot$$

$$\cdot\, q^{n\mu+\lambda(\lambda-1)+\mu(\mu-1)} \cdot \Phi^{g:h;H}_{l:k;K} \begin{bmatrix} a_g q^{\lambda+\mu} : b_h q^\mu;\, B_H q^\lambda; \\ \qquad\qquad q^\lambda x,\, y \\ \alpha_l q^{\lambda+\mu} : c_k q^\mu;\, C_K q^\lambda; \end{bmatrix} \tag{14}$$

$$=\sum_{\mu=0}^{n}\sum_{\lambda=0}^{n-\mu}\begin{bmatrix} n \\ \mu \end{bmatrix}\begin{bmatrix} n-\mu \\ \lambda \end{bmatrix}\cdot\frac{\Pi[a_g]_{\lambda+\mu}\,\Pi[b_h]_\mu\,\Pi[B_H]_\lambda\, a^{\lambda+\mu}\, y^\lambda\, x^\mu}{\Pi[\alpha_l]_{\lambda+\mu}\,\Pi[c_k]_\mu\,\Pi[C_K]_\lambda\,[a]_{\lambda+\mu}}\cdot$$

$$q^{n\lambda+\lambda(\lambda-1)+\mu(\mu-1)}\;\Phi\begin{matrix} g:\ h;\ H \\ \\ l:\ k;\ K\end{matrix}\begin{bmatrix} a_g\, q^{\lambda+\mu}:\ b_h\, q^\mu;\ B_H\, q^\lambda; & \\ & x;\ yq^\mu \\ \alpha_l\, q^{\lambda+\mu}:\ c_k\, q^\mu;\ C_K\, q^\lambda & \end{bmatrix} \tag{15}$$

$$\Phi\begin{matrix} g:\ h;\ H+1 \\ \\ l:\ k;\ K+1\end{matrix}\begin{bmatrix} a_g:\ b_h;\ B_H;\ aq^{-n}; & \\ & x,\ y \\ \alpha_l:\ c_k;\ C_K;\ a & \end{bmatrix}$$

$$=\sum_{\lambda=0}^{n}\begin{bmatrix} n \\ \lambda \end{bmatrix}\frac{\Pi[a_g]_\lambda\,\Pi[B_H]_\lambda\,(-ay)^\lambda\, q^{-n\lambda+\lambda(\lambda-1)/2}}{\Pi[\alpha_l]_\lambda\,\Pi[C_K]_\lambda\,\Pi[a]_\lambda}\cdot\Phi\begin{matrix} g:\ h;\ H+1 \\ l:\ k;\ K+1\end{matrix}\begin{bmatrix} a_g\, q^\lambda:\ b_h;\ B_H\, q^\lambda,\ a; & \\ & x,\ y \\ \alpha_l\, q^\lambda:\ c_k;\ C_K\, q^\lambda,\ aq^\lambda; & \end{bmatrix} \tag{16}$$

$$\Phi\begin{matrix} g:\ h+1;\ H+1 \\ \\ l:\ k+1;\ K+1\end{matrix}\begin{bmatrix} a_g:\ b_h,\ aq^{-m};\ B_H,\ Aq^{-n}; & \\ & x,\ y \\ \alpha_l:\ c_k,\ a;\ C_K,\ A; & \end{bmatrix}$$

$$=\sum_{\lambda=0}^{m}\sum_{\mu=0}^{n}\begin{bmatrix} m \\ \lambda \end{bmatrix}\begin{bmatrix} n \\ \mu \end{bmatrix}\frac{\Pi[a_g]_{\lambda+\mu}\,\Pi[b_h]_\lambda\,\Pi[B_H]_\mu\,(-ax)^\lambda\,(-Ay)^\mu}{\Pi[\alpha_l]_{\lambda+\mu}\,\Pi[c_k]_\lambda\,\Pi[C_K]_\mu\,[a]_\lambda\,[A]_\mu}\, q^{-n(\lambda+\mu)+[\lambda(\lambda-1)+\mu(\mu-1)]/2}\cdot$$

$$\cdot\,\Phi\begin{matrix} g:\ h+1;\ H+1 \\ \\ l:\ k+1;\ K+1\end{matrix}\begin{bmatrix} a_g\ \ q^{\lambda+\mu}:\ b_h\, q^\lambda,\ a;\ B_H\, q^\mu,\ A; & \\ & x,\ y \\ \alpha_l\, q^{\lambda+\mu}:\ c_k\, q^\lambda,\ aq^\lambda;\ C_K\, q^\mu;\ Aq^\mu & \end{bmatrix} \tag{17}$$

$$\Phi\begin{matrix} g:\ h+1;\ H+1 \\ \\ l:\ k+1;\ K+1\end{matrix}\begin{bmatrix} a_g:\ b_h;\ aq^m;\ B_H,\ Aq^{-n}; & \\ & x,\ y \\ \alpha_l:\ c_k,\ \ a;\ \ C_K,\ \ A; & \end{bmatrix}$$

$$=\sum_{\lambda=0}^{m}\sum_{\mu=0}^{n}\begin{bmatrix} m \\ \lambda \end{bmatrix}\begin{bmatrix} n \\ \mu \end{bmatrix}\frac{\Pi[a_g]_{\lambda+\mu}\,\Pi[b_h]_\lambda\,\Pi[B_H]_\mu\,(ax)^\lambda\,(-ay)^\mu}{\Pi[\alpha_l]_{\lambda+\mu}\,\Pi[c_k]_\lambda\,\Pi[C_K]_\mu\,[a]_\lambda\,[A]_\mu}\cdot q^{-n\mu+[\lambda(\lambda-1)+\mu(\mu-1)]/2}\cdot$$

$$\cdot\,\Phi\begin{matrix} g:\ h;\ H+1 \\ \\ l:\ k;\ K+1\end{matrix}\begin{bmatrix} a_g\ q^{\lambda+\mu}:\ b_h\ q^\lambda;\ B_H\ q^\mu,\ A; & \\ & x,\ y \\ \alpha_l\, q^{\lambda+\mu};\ c_k\, q^\lambda;\ C_K\, q^\mu,\ Aq^n; & \end{bmatrix} \tag{18}$$

***Proof*:**

On writing the value of left hand side of (12) as in (3), we have

$$\Phi\begin{matrix} g:\ h;\ H+1 \\ \\ l:\ k;\ K+1\end{matrix}\begin{bmatrix} a_g:b_h;\ B_H,\ aq^n; & \\ & x,\ y \\ \alpha_l:\ c_k;\ C_K,\ a; & \end{bmatrix}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\Pi[a_g]_{r+s}\,\Pi[b_h]_r\,\Pi[B_H]_s\,[aq^n]_s\, x^r\, y^s}{\Pi[\alpha_l]_{r+s}\,\Pi[c_k]_r\,\Pi[C_K]_s\,[a]_s\,[q]_r\,[q]_s}$$

$$=\sum_{r=0}^{\infty}\frac{\Pi[a_g]_r\,\Pi[b_h]_r\, x^r}{\Pi[\alpha^l]\,\Pi[c_k]_r\,[q]_r}\;{}_{g+H+1}\Phi_{l+K+1}\,(a_g\, q^r,\ B_H,\ aq^n;\ \alpha_l\, q^r,\ C_J,\ a;\ y)$$

Using (9) and (3), (12) follows immediately.

Other results also follow from the similar manupulations.

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